

Symplectic actions of two tori on four manifolds

Alvaro Pelayo*

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Abstract

We classify symplectic actions of 2-tori on compact, connected symplectic 4-manifolds, up to equivariant symplectomorphisms. This extends results of Atiyah, Guillemin–Sternberg, Delzant and Benoist. The classification is in terms of a collection of invariants, which are invariants of the topology of the manifold, of the torus action and of the symplectic form. We construct explicit models of such symplectic manifolds with torus actions, defined in terms of these invariants.

We also classify, up to equivariant symplectomorphisms, symplectic actions of $(2n - 2)$ -dimensional tori on $2n$ -dimensional symplectic manifolds, when at least one orbit is a $(2n - 2)$ -dimensional symplectic submanifold. Then we show that a $2n$ -dimensional symplectic manifold (M, σ) equipped with a free symplectic action of a $(2n - 2)$ -dimensional torus with at least one $(2n - 2)$ -dimensional symplectic orbit is equivariantly diffeomorphic to $M/T \times T$ equipped with the translational action of T . Thus two such symplectic manifolds are equivariantly diffeomorphic if and only if their orbit spaces are surfaces of the same genus.

1 Introduction

We extend the theory of Atiyah [1], Guillemin [19], Guillemin–Sternberg [18], Delzant [10], and Benoist [3] to symplectic actions of tori which are not necessarily Hamiltonian. Although Hamiltonian actions of n -dimensional tori on $2n$ -dimensional manifolds are present in many integrable systems in classical mechanics, non-Hamiltonian actions occur also in physics, c.f. Novikov’s article [40]. Interest on non-Hamiltonian motions may be found in the physics literature, for example: Sergi–Ferrario [48], Tarasov [58] and Tuckerman’s articles [56], [57] and the references therein.

In this paper we give a classification of symplectic actions of 2-tori on compact connected symplectic 4-manifolds in terms of a collection of invariants, some of which are algebraic while others are topological or geometric. A consequence of our classifications is the following.

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Theorem 1.1. *The only 4-dimensional symplectic manifold equipped with a non-locally-free and non-Hamiltonian effective symplectic action of a 2-torus is, up to equivariant symplectomorphisms, the product $\mathbb{T}^2 \times S^2$, where $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$ and the first factor of \mathbb{T}^2 acts on the left factor by translations on one component, and the second factor acts on S^2 by rotations about the vertical axis of S^2 . The symplectic form is a positive linear combination of the standard translation invariant form on \mathbb{T}^2 and the standard rotation invariant form on S^2 .*

Duistermaat and I showed in [12] that a compact connected symplectic manifold (M, σ) with a symplectic torus action with at least one coisotropic principal orbit is an associated G -bundle $G \times_H M_h$ whose fiber is a symplectic toric manifold M_h with T_h -action and whose base G/H is a torus bundle over a torus. Here T_h is the unique maximal subtorus of T which acts in a Hamiltonian fashion on M , G is a two-step nilpotent Lie group which is an extension of the torus T , and H is a commutative closed Lie subgroup of G which acts on M_h via T_h and is defined in terms of the holonomy of a certain connection for the principal bundle $M_{\text{reg}} \rightarrow M_{\text{reg}}/T$, where M_{reg} is the set of points where the action is free. Precisely, $G = T \times (\mathfrak{l}/\mathfrak{t}_h)^*$ where \mathfrak{l} is the kernel of the antisymmetric bilinear form σ^t on \mathfrak{t} which gives the restriction of σ to the orbits, and \mathfrak{t}_h is the Lie algebra of the torus T_h . The additive group $(\mathfrak{l}/\mathfrak{t}_h)^* \subset \mathfrak{l}^*$, viewed as the set of linear forms on \mathfrak{l} which vanish on \mathfrak{t}_h , is the maximal subgroup of \mathfrak{t}^* which acts on the orbit space M/T . We then proved a precise version of the following.

Rough Classification 1.2 (Duistermaat–P.-, 2005). *Symplectic actions of tori on compact connected symplectic manifolds with coisotropic principal orbits are classified by an antisymmetric bilinear form σ^t on \mathfrak{t} (the restriction of σ to the orbits), the Hamiltonian torus T_h , the momentum polytope associated to T_h by the Atiyah–Guillemin–Sternberg theorem, a discrete cocompact subgroup in $(\mathfrak{l}/\mathfrak{t}_h)^* \subset \mathfrak{t}^*$ (the period lattice of $(\mathfrak{l}/\mathfrak{t}_h)^*$), an antisymmetric bilinear form $c: (\mathfrak{l}/\mathfrak{t}_h)^* \times (\mathfrak{l}/\mathfrak{t}_h)^* \rightarrow \mathfrak{l}$ with certain integrality properties (represents the Chern class of the principal T -bundle $M_{\text{reg}} \rightarrow M_{\text{reg}}/T$), and the holonomy invariant of a so called admissible connection for the principal bundle $M_{\text{reg}} \rightarrow M_{\text{reg}}/T$.*

On the other hand suppose that (M, σ) is a compact, connected $2n$ -dimensional symplectic manifold equipped with an effective action of a $(2n - 2)$ -dimensional torus T for which at least one T -orbit is a $(2n - 2)$ -dimensional symplectic submanifold of (M, σ) . Then the orbit space M/T is a compact, connected, smooth, orientable orbisurface (2-dimensional orbifold) and the projection mapping $\pi: M \rightarrow M/T$ is a smooth principal T -bundle for which the collection $\{(T_x(T \cdot x))^{\sigma_x}\}_{x \in M}$ of symplectic orthogonal complements to the tangent spaces to the T -orbits is a flat connection. Let p_0 be any regular point in M/T , $\pi_1^{\text{orb}}(M/T, p_0)$ be the orbifold fundamental group, and $\widetilde{M/T}$ be the orbifold universal cover of M/T . Then the symplectic manifold (M, σ) is T -equivariantly symplectomorphic to the associated bundle $\widetilde{M/T} \times_{\pi_1^{\text{orb}}(M/T, p_0)} T$, where $\pi_1^{\text{orb}}(M/T, p_0)$ acts on T by means of the monodromy homomorphism of the aforementioned flat connection. We will describe the symplectic form on this space, as well as the torus action, in Definition 2.32. The T -action on this bundle comes from the T -action on $\widetilde{M/T} \times T$ by translations on the right factor. Then we prove a precise version of the following, c.f. Theorem 4.33:

Rough Classification 1.3. *Symplectic actions of $(2n - 2)$ -dimensional tori on compact connected symplectic $2n$ -manifolds for which at least one T -orbit is a $(2n - 2)$ -dimensional symplectic submanifold are classified by a non-degenerate antisymmetric bilinear form σ^t on \mathfrak{t} (the restriction of σ to the orbits), the Fuchsian signature of the orbit space M/T , which is a compact, connected orbisurface (and by this we mean the genus g of the underlying surface and the tuple of orders \vec{o} of the orbifold singularities of M/T), the total symplectic area of M/T , and an element in T^{2g+n}/\mathcal{G} which encodes the holonomy of the aforementioned flat connection for $\pi: M \rightarrow M/T$, where n is the number of orbifold singular points of M/T , and where \mathcal{G} is the group of matrices*

$$\mathcal{G} := \left\{ \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \in \mathrm{GL}(2g + n, \mathbb{Z}) \mid A \in \mathrm{Sp}(2g, \mathbb{Z}), D \in \mathcal{MS}_n^{\vec{o}} \right\}. \quad (1.1)$$

Here $\mathrm{Sp}(2g, \mathbb{Z})$ stands for symplectic matrices and $\mathcal{MS}_n^{\vec{o}}$ for permutation matrices which preserve the tuple \vec{o} of orbifold singularities of M/T .

Moreover we show that if the T -action is free, then M is T -equivariantly diffeomorphic to the product $M/T \times T$, equipped with the action of T by translations on the right factor. Thus two such symplectic manifolds are equivariantly diffeomorphic if and only if their corresponding orbit spaces are surfaces of the same genus. Using the rough classifications 1.2, 1.3 as a stepping stone we obtain the following classification, a precise and explicit statement of which is Theorem 5.4 (pp. 59).

Rough Classification 1.4. *Let (M, σ) be a compact connected 4-dimensional symplectic manifold equipped with an effective symplectic action of a 2-torus T . Then one and only one of the following cases occurs:*

- 1) (M, σ) is a 4-dimensional symplectic toric manifold, determined by its Delzant polygon.
- 2) (M, σ) is equivariantly symplectomorphic to a product $\mathbb{T}^2 \times S^2$, where $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$ and the first factor of \mathbb{T}^2 acts on the left factor by translations on one component, and the second factor acts on S^2 by rotations about the vertical axis of S^2 . The symplectic form is a positive linear combination of the standard translation invariant form on \mathbb{T}^2 and the standard rotation invariant form on S^2 .
- 3) T acts freely on (M, σ) with all T -orbits being Lagrangian 2-tori, and it is classified in (rough) classification 1.2 by a discrete cocompact subgroup P of \mathfrak{t}^* , an antisymmetric bilinear mapping $c: \mathfrak{t}^* \times \mathfrak{t}^* \rightarrow \mathfrak{t}$ which satisfies certain symmetry and integrality properties, and the so called holonomy invariant of a so called admissible connection for $M_{\mathrm{reg}} \rightarrow M_{\mathrm{reg}}/T$.
- 4) T acts on (M, σ) with all T -orbits being symplectic 2-tori, and it is classified in (rough) classification 1.3 by an antisymmetric bilinear form σ^t on \mathfrak{t} , the Fuchsian signature of M/T , the total symplectic area of M/T , and an element in T^{2n+g}/\mathcal{G} , where g is the genus of M/T , n is the number of singular points of M/T , and \mathcal{G} is the group given by (1.1).

The paper is organized as follows. In Section 2 we describe models of (M, σ) up to T -equivariant symplectomorphisms and up to T -equivariant diffeomorphisms.

In Section 3 we classify free symplectic torus actions of a $(2n - 2)$ -dimensional torus T on $2n$ -dimensional symplectic manifolds, when at least one T -orbit is a $(2n - 2)$ -dimensional symplectic submanifold of (M, σ) , up to T -equivariant symplectomorphisms, c.f. Theorem 3.15, and also up to T -equivariant diffeomorphisms, c.f. Corollary 3.16.

In Section 4 we extend the results in Section 3 to non-free actions. We only present those parts of the proofs which are different from the proofs in Section 3, and hence we suggest that the paper be read linearly between Section 3 and Section 5, both included. In turn this approach has the benefit that with virtually no repetition we are able to present the classification in the free case in terms of invariants which are easier to describe than in the general case.

In Section 5 we provide a classification of symplectic actions of 2-dimensional tori on compact, connected, symplectic 4-dimensional manifolds. This generalizes the 4-dimensional Delzant's theorem [10] to non-Hamiltonian actions.

There is extensive literature on the classification of Hamiltonian, symplectic and/or smooth torus actions. The papers closest to our paper in spirit are the paper [10] by Delzant on the classification of symplectic-toric manifolds, and the paper by Duistermaat and the author [12] on the classification of symplectic torus actions with coisotropic principal orbits. The following are other contributions related to our work. The paper [26] by Karshon on the classification of Hamiltonian circle actions on compact connected symplectic 4-manifolds. The book of Audin's [2] on Hamiltonian torus actions, and Orlik-Raymond's [41] and Pao's [45] papers, on the classification of actions of 2-dimensional tori on 4-dimensional compact connected smooth manifolds – they do not assume an invariant symplectic structure. Kogan [30] studied completely integrable systems with local torus actions. Karshon and Tolman studied centered complexity one Hamiltonian torus actions in arbitrary dimensions in their article [28] and Hamiltonian torus actions with 2-dimensional symplectic quotients in [27]. McDuff [36] and McDuff and Salamon [37] studied non-Hamiltonian circle actions, and Ginzburg [15] non-Hamiltonian symplectic actions of compact groups under the assumption of a “Lefschetz condition”. Symington [50] and Leung and Symington [31] classified 4-dimensional compact connected symplectic manifolds which are fibered by Lagrangian tori where the fibration may have elliptic or focus-focus singularities.

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2 Global model

Unless otherwise stated we assume throughout the section that (M, σ) is a smooth compact and connected symplectic manifold and T is a torus which acts effectively on (M, σ) by means of symplectomorphisms. We furthermore assume that at least one T -orbit is a $\dim T$ -dimensional symplectic submanifold of (M, σ) .

2.1 The orbit space M/T

We describe the structure of the orbit space M/T , c.f. Definition 2.13.

2.1.1 Symplectic form on the T -orbits

We prove that the symplectic form on every T -orbit of M is given by the same non-degenerate antisymmetric bilinear form.

Let X be an element of the Lie algebra \mathfrak{t} of T , and denote by X_M the smooth vector field on M obtained as the infinitesimal action of X on M . Let ω be a smooth differential form, let L_v denote the Lie derivative with respect to a vector field v , and let $i_v \omega$ denote the usual inner product of ω with v . Since the symplectic form σ is T -invariant, we have that $d(i_{X_M} \sigma) = L_{X_M} \sigma = 0$, where the first equality follows by combining $d\sigma = 0$ and the homotopy identity $L_v = d \circ i_v + i_v \circ d$. The following result follows from [12, Lem. 2.1].

Lemma 2.1. *Let (M, σ) be a compact, connected, symplectic manifold equipped with an effective symplectic action of a torus T for which there is at least one T -orbit which is a $\dim T$ -dimensional symplectic submanifold of (M, σ) . Then there exists a unique non-degenerate antisymmetric bilinear form $\sigma^t: \mathfrak{t} \times \mathfrak{t} \rightarrow \mathbb{R}$ on the Lie algebra \mathfrak{t} of T such that*

$$\sigma_x(X_M(x), Y_M(x)) = \sigma^t(X, Y), \quad (2.1)$$

for every $X, Y \in \mathfrak{t}$, and every $x \in M$.

Proof. In [12, Lem. 2.1] it was shown that there is a unique antisymmetric bilinear form $\sigma^t: \mathfrak{t} \times \mathfrak{t} \rightarrow \mathbb{R}$ on the Lie algebra \mathfrak{t} of T such that expression (2.1) holds for every $X, Y \in \mathfrak{t}$, and every $x \in M$. We recall the proof which was given in [12]. It follows from Benoist², [3, Lem. 2.1] that if u and v

¹The question treated in this paper was raised by professor M. Symington and was communicated to the author by professor Y. Karshon.

²Although the observation was probably first done by Kostant. I thank Y. Karshon for making me aware of this.

are smooth vector fields on M such that $L_u\sigma = 0$ and $L_v\sigma = 0$, then $[u, v] = \text{Ham}_{\sigma(u, v)}$. Indeed, observe that

$$i_{[u, v]}\sigma = L_u(i_v\sigma) = i_u(d(i_v\sigma)) + d(i_u(i_v\sigma)) = d(\sigma(u, v)). \quad (2.2)$$

Here we used $L_u\sigma = 0$ in the first equality, the homotopy formula for the Lie derivative in the second identity, and finally $d\sigma = 0$, the homotopy identity and $L_v\sigma = 0$ in the third equality. Applying (2.2) to $u = X_M, v = Y_M$, where $X, Y \in \mathfrak{t}$, we obtain that

$$i_{[X_M, Y_M]}\sigma = d(\sigma(X_M, Y_M)). \quad (2.3)$$

On the other hand, since T is commutative,

$$[X_M, Y_M] = -[X, Y]_M = 0, \quad (2.4)$$

and hence $[X, Y] = 0$. Combining expression (2.3) with expression (2.4), we obtain that the derivative of the real valued function $x \mapsto \sigma_x(X_M(x), Y_M(x))$ identically vanishes on M , which in virtue of the connectedness of M and the fact that σ is a symplectic form, implies expression (2.1) for a certain antisymmetric bilinear form σ^t in \mathfrak{t} . Since there is a T -orbit of dimension $\dim T$ which is a symplectic submanifold of (M, σ) , the form σ^t must be non-degenerate. \square

Remark 2.2 Each tangent space $T_x(T \cdot x)$ equals the linear span of the vectors $X_M(x), X \in \mathfrak{t}$. The collection of tangent spaces $T_x(T \cdot x)$ to the T -orbits $T \cdot x$ forms a smooth $\dim T$ -dimensional distribution³, which is integrable, where the integral manifold through x is precisely the T -orbit $T \cdot x$. Since the $X_M, X \in \mathfrak{t}$, are T -invariant vector fields, the distribution $H = \{T_x(T \cdot x)\}_{x \in M}$ is T -invariant. Each element of H is a symplectic vector space. \oslash

2.1.2 Stabilizer subgroup classification

Recall that if M is an arbitrary smooth manifold equipped with a smooth action of a torus T , for each $x \in M$ we write $T_x := \{t \in T \mid t \cdot x = x\}$ for the *stabilizer subgroup of the action of T on M at the point x* . T_x is a closed Lie subgroup of T . In this section we study the stabilizer subgroups of the action of T on (M, σ) .

In [12, Sec. 2], Duistermaat and the author pointed out that for general symplectic torus actions the stabilizer subgroups of the action need not be connected, which is in contrast with the symplectic actions whose principal orbits are Lagrangian submanifolds, where the stabilizer subgroups are subtori of T ; such fact also may be found as statement (1)(a) in Benoist's article [3, Lem. 6.7].

Lemma 2.3. *Let T be a torus. Let (M, σ) be a compact, connected, symplectic manifold equipped with an effective, symplectic action of the torus T , such that at least one T -orbit is a $\dim T$ -dimensional symplectic submanifold of (M, σ) . Then the stabilizer subgroup of the T -action at every point in M is a finite abelian group.*

³Since T is a commutative group, the Lie brackets of X_M and Y_M are zero for all $X, Y \in \mathfrak{t}$, which implies that \mathcal{D} is integrable, which in particular verifies the integrability theorem of Frobenius [59, Th. 1.60] for our particular assumptions (although we do *not* need it).

Proof. Let \mathfrak{t}_x denote the Lie algebra of the stabilizer subgroup T_x of the action of T on M at the point x . In the article of Duistermaat and the author [12, Lem. 2.2], we observed that for every $x \in M$ there is an inclusion $\mathfrak{t}_x \subset \ker \sigma^t$, and since by Lemma 2.1 σ^t is non-degenerate, its kernel $\ker \sigma^t$ is trivial, which in turn implies that \mathfrak{t}_x is the trivial vector space, and hence T_x , which is a closed and hence compact subgroup of T , must be a finite group. \square

Lemma 2.4. *Let T be a torus. Let (M, σ) be a compact, connected, symplectic manifold equipped with an effective, symplectic action of the torus T such that at least one T -orbit is a $\dim T$ -dimensional symplectic submanifold of (M, σ) . Then every T -orbit is a $\dim T$ -dimensional symplectic submanifold of (M, σ) .*

Proof. Let $x \in M$. Since the torus T is a compact group, the action of T on the smooth manifold M is proper, and the mapping

$$t \mapsto t \cdot x: T/T_x \rightarrow T \cdot x \quad (2.5)$$

is a diffeomorphism, c.f. [16, Appendix B] or [8, Sec. 23.2], and in particular, the dimension of quotient group T/T_x equals the dimension of the T -orbit $T \cdot x$. Since by Lemma 2.3 each stabilizer subgroup T_x is finite, the dimension of T/T_x equals $\dim T$, and hence every T -orbit is $\dim T$ -dimensional. By Lemma 2.1 the symplectic form σ restricted to any T -orbit of the T -action is non-degenerate and hence $T \cdot x$ is a symplectic submanifold of (M, σ) . \square

Corollary 2.5. *Let (M, σ) be a compact connected symplectic manifold equipped with an effective symplectic action of a torus T for which at least one, and hence every T -orbit is a $\dim T$ -dimensional symplectic submanifold of (M, σ) . Then there exists only finitely many different subgroups of T which occur as stabilizer subgroups of the action of T on M , and each of them is a finite group.*

Proof. It follows from Lemma 2.3 that every stabilizer subgroup of the action of T on M is a finite group. It follows from the tube theorem, c.f. [11, Th. 2.4.1] or [16, Th. B24] that in the case of a compact smooth manifold equipped with an effective action of a torus T , there exists only finitely many different subgroups of T which occur as stabilizer subgroups. \square

If M is 4-dimensional and T is 2-dimensional we have the following stronger statement, which follows from the tube theorem, since a finite group acting linearly on a disk must be a cyclic group acting by rotations.

Lemma 2.6. *Let T be a 2-torus. Let (M, σ) be a compact, connected, symplectic 4-manifold equipped with an effective, symplectic action of T , such that at least one, and hence every T -orbit is a 2-dimensional symplectic submanifold of (M, σ) . Then the stabilizer subgroup of the T -action at every point in M is a cyclic abelian group.*

2.1.3 Orbifold structure of M/T

We denote the space of all orbits in M of the T -action by M/T , and by $\pi: M \rightarrow M/T$ the canonical projection which assigns to each $x \in M$ the orbit $T \cdot x$ through the point x . The orbit

space is provided with the maximal topology for which the canonical projection π is continuous; this topology is Hausdorff. Because M is compact and connected, the orbit space M/T is compact and connected. For each connected component C of an orbit type $M^H := \{x \in M \mid T_x = H\}$ in M of the subgroup H of T , the action of T on C induces a proper and free action of the torus T/H on C , and $\pi(C)$ has a unique structure of a smooth manifold such that $\pi: C \rightarrow \pi(C)$ is a principal T/H -bundle. The orbit space M/T is not in general a smooth manifold, c.f. Example 2.8. Our next goal is to show that the orbit space M/T has a natural structure of smooth orbifold.

Example 2.7 [Free action] Let (M, σ) be the Cartesian product $(\mathbb{R}/\mathbb{Z})^2 \times S^2$ equipped with the product symplectic form of the standard symplectic (area) form on the torus $(\mathbb{R}/\mathbb{Z})^2$ and the standard area form on the sphere S^2 . Let T be the 2-torus $(\mathbb{R}/\mathbb{Z})^2$, and let T act on M by translations on the left factor of the product. Such action of T on M is free, it has symplectic 2-tori as T -orbits, and the orbit space M/T is equal to the 2-sphere S^2 . Probably the simplest example of a 4-dimensional symplectic manifold equipped with a symplectic action of a 2-torus for which the torus orbits are symplectic 2-dimensional tori is the 4-dimensional torus $(\mathbb{R}/\mathbb{Z})^2 \times (\mathbb{R}/\mathbb{Z})^2$ with the standard symplectic form, on which the 2-dimensional torus $(\mathbb{R}/\mathbb{Z})^2$ acts by multiplications on two of the copies of \mathbb{R}/\mathbb{Z} inside of $(\mathbb{R}/\mathbb{Z})^4$. The orbit space is a 2-dimensional torus, so a smooth manifold. \circlearrowright

Example 2.8 [Non-free action] Consider the Cartesian product $S^2 \times (\mathbb{R}/\mathbb{Z})^2$ of the 2-sphere and the 2-torus equipped with the product symplectic form of the standard symplectic (area) form on the torus $(\mathbb{R}/\mathbb{Z})^2$ and the standard area form on the sphere S^2 . The 2-torus $(\mathbb{R}/\mathbb{Z})^2$ acts freely by translations on the right factor of the product $S^2 \times (\mathbb{R}/\mathbb{Z})^2$. Consider the action of the finite group $\mathbb{Z}/2\mathbb{Z}$ on S^2 which rotates each point horizontally by 180 degrees, and the action of $\mathbb{Z}/2\mathbb{Z}$ on the 2-torus $(\mathbb{R}/\mathbb{Z})^2$ given by the antipodal action on the first circle. The diagonal action of $\mathbb{Z}/2\mathbb{Z}$ on $S^2 \times (\mathbb{R}/\mathbb{Z})^2$ is free and hence the quotient space $S^2 \times_{\mathbb{Z}/2\mathbb{Z}} (\mathbb{R}/\mathbb{Z})^2$ is a smooth manifold. Let (M, σ) be this associated bundle $S^2 \times_{\mathbb{Z}/2\mathbb{Z}} (\mathbb{R}/\mathbb{Z})^2$ with the symplectic form and T -actions inherited from the ones given in the product $S^2 \times (\mathbb{R}/\mathbb{Z})^2$, where $T = (\mathbb{R}/\mathbb{Z})^2$. The action of T on M is not free and the T -orbits are symplectic 2-dimensional tori. The orbit space M/T is equal to $S^2/(\mathbb{Z}/2\mathbb{Z})$, which is a smooth orbifold with two singular points of order 2, the South and North poles of S^2 . \circlearrowright

There is no standard definition of orbifold, so we do not use terminology in orbifolds without clarifying or introducing it. Following largely but not entirely Boileau–Maillot–Porti [6, Sec. 2.1.1], we define orbifold. We also borrow from ideas in Satake [46], [47] and Thurston [54]. Our definition of orbifold is close to that in Haefliger’s paper [20, Sec. 4]. Unfortunately Haefliger’s paper does not appear to be so well known and is frequently not given proper credit. I thank Y. Karshon for making me aware of this.

Definition 2.9 A *smooth n-dimensional orbifold* \mathcal{O} is a metrizable topological space $|\mathcal{O}|$ endowed with an equivalence class of orbifold atlases for \mathcal{O} . An *orbifold atlas for* \mathcal{O} is a collection $\{(U_i, \tilde{U}_i, \phi_i, \Gamma_i)\}_{i \in I}$ where for each $i \in I$, U_i is an open subset of \mathcal{O} , \tilde{U}_i is an open and connected subset of \mathbb{R}^n , $\phi_i: \tilde{U}_i \rightarrow U_i$ is a continuous map, called an *orbifold chart*, and Γ_i is a finite group of diffeomorphisms of \tilde{U}_i , satisfying:

- i) the U_i 's cover \mathcal{O} ,
- ii) each ϕ_i factors through a homeomorphism between \tilde{U}_i/Γ_i and U_i , and
- iii) the charts are compatible. This means that for each $x \in \tilde{U}_i$ and $y \in \tilde{U}_j$ with $\phi_i(x) = \phi_j(y)$, there is a diffeomorphism ψ between a neighborhood of x and a neighborhood of y such that $\phi_j(\psi(z)) = \phi_i(z)$ for all z in such neighborhood.

Two orbifold atlases are *equivalent* if their union is an orbifold atlas. If $x \in U_i$, the *local group* Γ_x of \mathcal{O} at a point $x \in \mathcal{O}$ is the isomorphism class of the stabilizer of the action of Γ_i on \tilde{U}_i at the point $\phi_i^{-1}(x)$. A point $x \in \mathcal{O}$ is *regular* if Γ_x is trivial, and *singular* otherwise. The *singular locus* is the set $\Sigma_{\mathcal{O}}$ of singular points of \mathcal{O} . We say that the orbifold \mathcal{O} is compact (resp. connected) if the topological space $|\mathcal{O}|$ is compact (resp. connected). An *orientation for an orbifold atlas* for \mathcal{O} is given by an orientation on each \tilde{U}_i which is preserved by every change of chart map ψ as in part iii) above. The orbifold \mathcal{O} is *orientable* if it has an orientation. \circlearrowright

Remark 2.10 One can replace metrizable in Definition 2.9 by Hausdorff which although is a weaker condition, it suffices for our purposes. \circlearrowright

By the tube theorem, c.f. [11, Th. 2.4.1], [16, Th. B24] for each $x \in M$ there exists a T -invariant open neighborhood U_x of the T -orbit $T \cdot x$ and a T -equivariant diffeomorphism Φ_x from U_x onto the associated bundle $T \times_{T_x} D_x$, where D_x is an open disk centered at the origin in $\mathbb{R}^k = \mathbb{C}^{k/2}$, where $k := \dim M - \dim T$, and T_x acts by linear transformations on D_x . The action of T on $T \times_{T_x} D_x$ is induced by the action of T by translations on the left factor of $T \times D_x$. Because Φ_x is a T -equivariant diffeomorphism, it induces a homeomorphism $\widehat{\Phi}_x$ on the quotient $\widehat{\Phi}_x: D_x/T_x \rightarrow \pi(U_x)$, and there is a commutative diagram of the form

$$\begin{array}{ccccc} T \times D_x & \xrightarrow{\pi_x} & T \times_{T_x} D_x & \xrightarrow{\Phi_x} & U_x \\ i_x \uparrow & & p_x \downarrow & & \downarrow \pi|_{U_x} \\ D_x & \xrightarrow{\pi'_x} & D_x/T_x & \xrightarrow{\widehat{\Phi}_x} & \pi(U_x) \end{array}, \quad (2.6)$$

where π_x, π'_x, p_x are the canonical projection maps, and i_x is the inclusion map. Let

$$\phi_x := \widehat{\Phi}_x \circ \pi'_x. \quad (2.7)$$

The word lift in Lemma 2.11 below is used in the sense that

$$\pi_{\Gamma'} \circ i' \circ F = f \circ \pi_{\Gamma} \circ i \quad (2.8)$$

where $i: D \rightarrow T \times D$, $i: D' \rightarrow T \times D'$ are the inclusion maps and $\pi_{\Gamma}: T \times D \rightarrow T \times_{\Gamma} D$, $\pi_{\Gamma'}: T \times D' \rightarrow T \times_{\Gamma'} D'$ are the canonical projections.

Lemma 2.11. *Let T be a torus. Let Γ, Γ' be finite subgroups of T respectively acting linearly on Γ, Γ' -invariant open subsets $D, D' \subset \mathbb{R}^m$. Let $z \in D, z' \in D'$. Let Γ, Γ' act on $T \times D, T \times D'$,*

respectively, by the diagonal action, giving rise to smooth manifolds $T \times_{\Gamma} D$ and $T \times_{\Gamma'} D'$ equipped with the T -actions induced by the action of T by left translations on $T \times D$, $T \times D'$, respectively. Let $f: T \times_{\Gamma} D \rightarrow T \times_{\Gamma'} D'$ be a T -equivariant diffeomorphism such that $f(T \cdot [1, z]_{\Gamma}) = T \cdot [1, z']_{\Gamma'}$. Then there exist open neighborhoods $U \subset D$ of z , and $U' \subset D'$ of z' , and a diffeomorphism $F: U \rightarrow U'$ which lifts f as in (2.8) and such that $F(z) = z'$.

One can obtain Lemma 2.11 applying the idea of the proof method of [22, Lem. 23] by replacing the mapping $f: \mathbb{R}^n/\Gamma \rightarrow U'/\Gamma'$ therein by $f: T \times_{\Gamma} D \rightarrow T \times_{\Gamma'} D'$.

Proposition 2.12. *Let T be a torus. Let (M, σ) be a compact, connected, symplectic manifold equipped with an effective, symplectic action of the torus T for which at least one, and hence every T -orbit, is a $\dim T$ -dimensional symplectic submanifold of (M, σ) . Then the collection of charts*

$$\widehat{\mathcal{A}} := \{(\pi(U_x), D_x, \phi_x, T_x)\}_{x \in M} \quad (2.9)$$

is an orbifold atlas for the orbit space M/T , where for each $x \in M$ the mapping ϕ_x is defined by expression (2.7), and the mappings $\widehat{\Phi}_x, \pi'_x$ are defined by diagram (2.6).

Proof. Because $x \in U_x$, we have that $\bigcup_{x \in M} U_x = M$, so the collection $\{\pi(U_x)\}_{x \in M}$ covers M/T . Since U_x is open, $\pi(U_x)$ is open, for each $x \in M$. Let $k := \dim M - \dim T$. By Lemma 2.3 the stabilizer group T_x is a finite group of diffeomorphisms. The disks D_x , $x \in M$, given by the tube theorem, are open subsets of \mathbb{R}^k , since T_x is a 0-dimensional subgroup of T . Because it is obtained as a composite of continuous maps, ϕ_x in (2.7) is continuous and it factors through $\widehat{\Phi}_x: D_x/T_x \rightarrow \pi(U_x)$, which is the homeomorphism on the bottom part of the right square of diagram (2.6).

It is left to show that the mappings ϕ_x, ϕ_y , where $x, y \in M$, are compatible on their overlaps. Indeed, pick $z \in D_x, z' \in D_y$ and assume that

$$\phi_x(z) = \phi_y(z'). \quad (2.10)$$

Let U_z be an open neighborhood of z such that $\widehat{\Phi}_x(U_z/T_x)$ is contained in $\pi(U_x) \cap \pi(U_y)$. Let $U_{z'} \subset D_y$ be the unique open subset of D_y such that $(\widehat{\Phi}_y)^{-1}(\widehat{\Phi}_x(U_z/T_x)) = U_{z'}/T_y$. Then the composite map

$$\widehat{\Psi}_{xy} := (\widehat{\Phi}_y)^{-1} \circ \widehat{\Phi}_x: U_z/T_x \rightarrow U_{z'}/T_y \quad (2.11)$$

is a homeomorphism which by (2.10) satisfies

$$\widehat{\Psi}_{xy}([z]_{T_x}) = [z']_{T_y}. \quad (2.12)$$

Since by the tube theorem the mappings $\Phi_x: T \times_{T_x} D_x \rightarrow U_x$ and $\Phi_y: T \times_{T_y} D_y \rightarrow U_y$ are T -equivariant diffeomorphisms, and by definition of p_x, p_y we have that $(p_x)^{-1}(U_z/T_x) = T \times_{T_x} U_z$ and $(p_y)^{-1}(U_{z'}/T_y) = T \times_{T_y} U_{z'}$, the composite map

$$\Psi_{xy} := (\Phi_y)^{-1} \circ \Phi_x: T \times_{T_x} U_z \rightarrow T \times_{T_y} U_{z'}, \quad (2.13)$$

is a T -equivariant diffeomorphism. By the commutativity of diagram (2.6), the map Ψ_{xy} lifts the map $\widehat{\Psi}_{xy}$. Then by (2.12), we have that

$$\Psi_{xy}(T \cdot [1, z]_{T_x}) = T \cdot [1, z']_{T_y}, \quad (2.14)$$

where $\pi_x: T \times D_x \rightarrow T \times_{T_x} D_x$, $\pi_y: T \times D_y \rightarrow T \times_{T_y} D_y$ are the canonical projection maps. Then the map in (2.13) is of the form in Lemma 2.11 and we can use this lemma to conclude that Ψ_{xy} lifts to a diffeomorphism $\psi_{xy}: W_z \rightarrow W_{z'}$, where $W_z \subset U_z \subset D_x$, and $W_{z'} \subset U_{z'} \subset D_y$ are open neighborhoods of z , z' respectively, and

$$\psi_{xy}(z) = z'. \quad (2.15)$$

Because the map (2.13) lifts the map (2.11), the diffeomorphism ψ_{xy} lifts the restricted homeomorphism $\widehat{\Psi}_{xy}: W_z/T_x \rightarrow W_{z'}/T_y$ induced by (2.11). Then by (2.7),

$$\phi_x \circ \psi_{xy} = \phi_y \quad (2.16)$$

on W_z . Expressions (2.15) and (2.16) and precisely describe the compatibility condition of the charts ϕ_x , ϕ_y on their overlaps. \square

Definition 2.13 Let T be a torus. Let (M, σ) be a compact, connected, symplectic manifold equipped with an effective, symplectic action of T for which at least one, and hence every T -orbit is a dim T -dimensional symplectic submanifold of (M, σ) . We call \mathcal{A} the class of atlases equivalent to the orbifold atlas $\widehat{\mathcal{A}}$ defined by expression (2.9) in Proposition 2.12. We denote the orbifold M/T endowed with the class \mathcal{A} by M/T , and the class \mathcal{A} is assumed. \oslash

Remark 2.14 Since M is compact and connected, M/T is compact and connected. If $\dim T = 2n - 2$, M/T is a compact, connected orbisurface. If the T action is free, then the local groups T_x in Definition 2.13 are all trivial, and M/T is a compact connected surface determined up to diffeomorphism by a non-negative integer, its topological genus.

Moreover, since every T -orbit is a dim T -dimensional symplectic submanifold of (M, σ) , the action of the torus T on M is a locally free and non-Hamiltonian action. Indeed, by Lemma 2.3, the stabilizers are finite, hence discrete groups. On the other hand, if an action is Hamiltonian the T -orbits are isotropic submanifolds, so the T -action cannot be Hamiltonian. \oslash

2.2 A flat connection for the projection $M \rightarrow M/T$

In this section we prove that the projection $\pi: M \rightarrow M/T$ onto the orbifold M/T (c.f. Proposition 2.12 and Definition 2.13) is a smooth principal T -orbibundle. For such orbibundle there are notions of connection, and of flat connection, which extend the classical definition for bundles. Then we show that $\pi: M \rightarrow M/T$ comes endowed with a flat connection (c.f. Proposition 2.12).

Let \mathcal{O} , \mathcal{O}' be smooth orbifolds. An *orbifold diffeomorphism* $f: \mathcal{O} \rightarrow \mathcal{O}'$ is a homeomorphism at the level of topological spaces $|\mathcal{O}| \rightarrow |\mathcal{O}'|$ such that for every $x \in \mathcal{O}$ there are charts

$\phi_i: \tilde{U}_i \rightarrow U_i$, $x \in U_i$, and $\phi'_j: \tilde{U}'_j \rightarrow U'_j$ such that $f(U_i) \subset U'_j$ and the restriction $f|_{U_i}$ may be lifted to a diffeomorphism $\tilde{f}: \tilde{U}_i \rightarrow \tilde{U}'_j$ which is equivariant with respect to some homomorphism $\Gamma_i \rightarrow \Gamma'_j$.

Definition 2.15 Let T be a torus, let (X, ω) be a compact, connected symplectic manifold endowed with an effective symplectic action of T for which at least one, and hence every T -orbit is a $\dim T$ -dimensional symplectic submanifold of (X, ω) . Equip X/T with the orbifold structure in Definition 2.13. Let Y be a smooth manifold equipped with a smooth effective action of T .

A continuous surjective map $p: Y \rightarrow X/T$ is a *smooth principal T -orbibundle* if the action of T on Y preserves the fibers of p and acts freely and transitively on them, and for every $z \in X/T$ the following holds. Let $\pi(U_x)$, D_x , ϕ_x , T_x as in (2.9). For each $x \in X$ for which $z \in \pi(U_x)$ there exists a T_x -invariant open subset \tilde{U}_x of D_x and a continuous map $\psi_x: T \times \tilde{U}_x \rightarrow Y$ which induces a diffeomorphism between $T \times_{T_x} \tilde{U}_x$ and $p^{-1}(\phi_x(\tilde{U}_x))$ such that $p \circ \psi_x = \phi_x \circ \pi_1$, where $\pi_1: T \times \tilde{U}_x \rightarrow \tilde{U}_x$ is the canonical projection. Here T_x acts on $T \times \tilde{U}_x$ by the diagonal action.

A *connection* for $p: Y \rightarrow X/T$ is a smooth vector subbundle H of the tangent bundle TY with the property that for each $y \in Y$, H_y is a direct complement in $T_y Y$ of the tangent space to the fiber of p that passes through y . We say that the connection H is *flat with respect to $p: Y \rightarrow X/T$* if the subbundle $H \subset TY$ is integrable considered as a smooth distribution on Y . \oslash

Remark 2.16 Let $p: Y \rightarrow X/T$ be a smooth principal T -orbibundle as in Definition 2.15. Then for every $y \in Y$ there are charts $\phi_i: \tilde{U}_i \rightarrow U_i$ of Y , $y \in U_i$, and $\phi'_j: \tilde{U}'_j \rightarrow U'_j$ of X/T , such that $p(U_i) \subset U'_j$ and the restriction $p|_{U_i}$ may be lifted to a smooth map $\tilde{p}: \tilde{U}_i \rightarrow \tilde{U}'_j$.

If the local group T_x is trivial for all $x \in X/T$, the action of T on X is trivial and X/T is a smooth manifold. Then the mapping $p: X \rightarrow X/T$ is a smooth principal T -bundle, in the usual sense of the theory of fiber bundles on manifolds, for example c.f. [49]. We also use the term T -bundle instead of T -orbibundle. \oslash

Proposition 2.17. Let (M, σ) be a compact connected symplectic manifold equipped with an effective symplectic action of a torus T for which at least one, and hence every T -orbit is a $\dim T$ -dimensional symplectic submanifold of (M, σ) . Then the collection $\Omega = \{\Omega_x\}_{x \in M}$ of subspaces $\Omega_x \subset T_x M$, where Ω_x is the σ_x -orthogonal complement to $T_x(T \cdot x)$ in $T_x M$, for every $x \in M$, is a smooth distribution on M . The projection mapping $\pi: M \rightarrow M/T$ is a smooth principal T -bundle of which Ω is a T -invariant flat connection.

Proof. For each $x \in M$ let $\tilde{U}_x = D_x$, where \tilde{U}_x , D_x are as in Definition 2.15. Let

$$\psi_x(t, z) := \Phi_x([t, z]_{T_x}),$$

where ψ_x is also the map in the definition. With these choices of \tilde{U}_x , ψ_x , by diagram (2.6) and by construction of the orbifold atlas on M/T , c.f. Proposition 2.12 and Definition 2.13, the projection mapping $\pi: M \rightarrow M/T$ is a smooth principal T -orbibundle. Because the tangent space to each T -orbit $(T_x(T \cdot x), \sigma|_{T_x(T \cdot x)})$ is a symplectic vector space, its symplectic orthogonal complement $(\Omega_x, \sigma|_{\Omega_x})$ is a symplectic vector space. Here $\sigma|_{\Omega_x}$, $\sigma|_{T_x(T \cdot x)}$ are the symplectic forms respectively

induced by σ on Ω_x , $T_x(T \cdot x)$. Consider the disk bundle $T \times_{T_x} \Omega_x$ where T_x acts by induced linearized action on Ω_x , and on $T \times \Omega_x$ by the diagonal action. The translational action of T on the left factor of $T \times \Omega_x$ descends to an action of T on $T \times_{T_x} \Omega_x$. There exists a unique T -invariant symplectic form σ' on $T \times_{T_x} \Omega_x$ such that if $\pi': T \times \Omega_x \rightarrow T \times_{T_x} \Omega_x$ is the canonical projection,

$$\pi'^* \sigma' = \sigma|_{T_x(T \cdot x)} \oplus \sigma|_{\Omega_x}, \quad (2.17)$$

where $\sigma|_{T_x(T \cdot x)} \oplus \sigma|_{\Omega_x}$ denotes the product symplectic form on $T \times \Omega_x$. Then by the symplectic tube theorem of Benoist [3, Prop. 1.9], Ortega and Ratiu [42] and Weinstein, which we use as it was formulated in [12, Sec. 11], there exists an open neighborhood D of $0 \in \Omega_x$, an open T -invariant neighborhood U of x in M , and a T -equivariant symplectomorphism $\Phi: T \times_{T_x} D \rightarrow U$ with $\Phi([1, 0]_{T_x}) = x$. By T -equivariance, Φ maps the zero section of $T \times_{T_x} \Omega_x$ to the T -orbit $T \cdot x$ through x . It follows from (2.17) that the symplectic–orthogonal complement to such section in $T \times_{T_x} \Omega_x$ is precisely the $(\dim M - \dim T)$ -dimensional manifold $\pi'(\{1\} \times \Omega_x)$, and hence, since Φ is a T -equivariant symplectomorphism, the image $\Phi(\pi'(\{1\} \times \Omega_x))$ is an integral manifold through x of dimension $\dim M - \dim T$, so $\{\Omega_x\}_{x \in M}$ is a T -invariant integrable distribution. \square

We can formulate Proposition 2.17 in the language of foliations as follows.

Corollary 2.18. *Let (M, σ) be a compact connected symplectic manifold equipped with an effective symplectic action of a torus T for which at least one, and hence every T -orbit is a $\dim T$ -dimensional symplectic submanifold of (M, σ) . Then the collection of integral manifolds to the symplectic orthogonal complements to the tangent spaces to the T -orbits, c.f. Proposition 2.17, is a smooth T -invariant $(\dim M - \dim T)$ -dimensional foliation of M .*

2.3 Symplectic Tube Theorem

It is convenient to state the the symplectic tube theorem used in the proof of Proposition 2.17 in a more standard way by replacing Ω_x by \mathbb{C}^m and $\sigma|_{\Omega_x}$ by $\sigma^{\mathbb{C}^m}$, respectively, where $m = 1/2(\dim M - \dim T)$. Indeed, write $i := \sqrt{-1} \in \mathbb{C}$ and let $\sigma^{\mathbb{C}^m}$ be the symplectic form on \mathbb{C}^m

$$\sigma^{\mathbb{C}^m} := \sum_{j=1}^m d\overline{z^j} \wedge dz^j / 2i. \quad (2.18)$$

Because Ω_x is a symplectic vector space, it has a symplectic basis with $2m$ elements which induces a direct sum decomposition of Ω_x into m mutually $\sigma|_{\Omega_x}$ -orthogonal two-dimensional linear subspaces E_j . The stabilizer group T_x acts by means of symplectic linear transformations on the symplectic vector space Ω_x , and the aforementioned symplectic basis can be chosen so that E_j is T_x -invariant. Averaging any inner product on each E_j over T_x , we obtain an T_x -invariant inner product β_j on E_j , which is unique if we also require that the symplectic inner product of any orthonormal basis with respect to $\sigma|_{\Omega_x}$ is equal to ± 1 . This leads to the existence of a unique complex structure on E_j such that, for any unit vector e_j in (E_j, β_j) , we have that e_j, ie_j is an orthonormal basis in (E_j, β_j) and $\sigma|_{\Omega_x}(e_j, ie_j) = 1$. This leads to an identification of E_j with \mathbb{C} , and hence of Ω_x with \mathbb{C}^m , with the symplectic form defined by (2.18). The element $c \in \mathbb{T}^m$ acts

on \mathbb{C}^m by sending $z \in \mathbb{C}^m$ to the element $c \cdot z$ such that $(c \cdot z)^j = c^j z^j$ for every $1 \leq j \leq m$. There is a unique monomorphism of Lie groups $\iota: T_x \rightarrow \mathbb{T}^m$ such that $h \in T_x$ acts on $\Omega_x = \mathbb{C}^m$ by sending $z \in \mathbb{C}^m$ to $\iota(h) \cdot z$, hence T_x acts on $T \times \mathbb{C}^m$ by

$$h \star (t, z) = (h^{-1}t, \iota(h)z). \quad (2.19)$$

Consider the disk bundle $T \times_{T_x} \mathbb{C}^m$ where T_x acts by (2.19). The translational action of T on $T \times \mathbb{C}^m$ descends to an action of T on $T \times_{T_x} \mathbb{C}^m$. By Lemma 2.1, the antisymmetric bilinear form $\sigma^t: \mathfrak{t} \times \mathfrak{t} \rightarrow \mathbb{R}$ is non-degenerate and hence it determines a unique symplectic form σ^T on T . In view of this, $\sigma|_{T_x(T \cdot x)} = \sigma^t$ in the proof of Proposition 2.17 does not depend on $x \in M$. The product symplectic form $\sigma^T \oplus \sigma^{\mathbb{C}^m}$ descends to a symplectic form on $T \times_{T_x} D$. With this terminology the proof of Proposition 2.17 implies the following.

Corollary 2.19 (Tube theorem for symplectic orbits). *Let (M, σ) be a compact, connected, symplectic manifold equipped with an effective symplectic action of a torus T for which at least one, and hence every T -orbit is a $\dim T$ -dimensional symplectic submanifold of (M, σ) . Then there is an open \mathbb{T}^m -invariant neighborhood D of $0 \in \mathbb{C}^m$, an open T -invariant neighborhood U of x in M , and a T -equivariant symplectomorphism $\Phi: T \times_{T_x} D \rightarrow U$ with $\Phi([1, 0]_{T_x}) = x$.*

In Corollary 2.19 the product symplectic form is defined pointwise as

$$(\sigma^T \oplus \sigma^{\mathbb{C}^m})_{(t, z)}((X, u), (X', u')) = \sigma^t(X, X') + \sigma^{\mathbb{C}^m}(u, u').$$

Here we identify each tangent space of the torus T with the Lie algebra \mathfrak{t} of T and each tangent space of a vector space with the vector space itself.

2.4 From Local to Global

We prove the main theorem of Section 2, Theorem 2.36, which gives a model description of (M, σ) up to T -equivariant symplectomorphisms.

2.4.1 Orbifold coverings of M/T

We start by recalling the notion of orbifold covering.

Definition 2.20 [6, Sec. 2.2] A *covering of a connected orbifold \mathcal{O}* is a connected orbifold $\tilde{\mathcal{O}}$ together with a continuous mapping $p: \tilde{\mathcal{O}} \rightarrow \mathcal{O}$, called an *orbifold covering map*, such that every point $x \in \mathcal{O}$ has an open neighborhood U with the property that for each component V of $p^{-1}(U)$ there is a chart $\phi: \tilde{V} \rightarrow V$ of $\tilde{\mathcal{O}}$ such that $p \circ \phi$ is a chart of \mathcal{O} . Two coverings $p_1: \mathcal{O}_1 \rightarrow \mathcal{O}$, $p_2: \mathcal{O}_2 \rightarrow \mathcal{O}$ are *equivalent* if there exists a diffeomorphism $f: \mathcal{O}_1 \rightarrow \mathcal{O}_2$ such that $p_2 \circ f = p_1$. A *universal cover of \mathcal{O}* is a covering $p: \hat{\mathcal{O}} \rightarrow \mathcal{O}$ such that for every covering $q: \tilde{\mathcal{O}} \rightarrow \mathcal{O}$, there exists a unique covering $r: \hat{\mathcal{O}} \rightarrow \tilde{\mathcal{O}}$ such that $q \circ r = p$. The *deck transformation group of a covering $p: \mathcal{O}' \rightarrow \mathcal{O}$* is the group of all self-diffeomorphisms $f: \mathcal{O}' \rightarrow \mathcal{O}'$ such that $p \circ f = p$, and it is denoted by $\text{Aut}(\mathcal{O}', p)$. \oslash

Remark 2.21 Assuming the terminology introduced in Definition 2.20, the open sets U, V equipped with the restrictions of the charts for $\widehat{\mathcal{O}}, \mathcal{O}$, are smooth orbifolds, and the restriction $p: V \rightarrow U$ is an orbifold diffeomorphism. \square

Lemma 2.22. *Let (M, σ) be a compact connected symplectic manifold equipped with an effective symplectic action of a torus T for which at least one, and hence every T -orbit is a $\dim T$ -dimensional symplectic submanifold of M , and let $x \in M$. Let $\pi: M \rightarrow M/T$ be the canonical projection, and let \mathcal{I}_x be the maximal integral manifold of the distribution of symplectic orthogonal complements to the tangent spaces to the T -orbits which goes through x (c.f. Proposition 2.17). Then the inclusion $i_x: \mathcal{I}_x \rightarrow M$ is an injective immersion between smooth manifolds and the composite $\pi \circ i_x: \mathcal{I}_x \rightarrow M/T$ is an orbifold covering map.*

Proof. Since Ω is a smooth distribution, the maximal integral manifold \mathcal{I}_x is injectively immersed in M , c.f. [59, Sec. 1]. Let $p \in M/T$, let $x \in \pi^{-1}(\{p\})$. Then $(\pi \circ i_x)^{-1}(\{p\}) = T \cdot x \cap \mathcal{I}_x$. By Corollary 2.19 there is an open \mathbb{T}^m -invariant neighborhood D_x of $0 \in \mathbb{C}^m$, an open T -invariant neighborhood U_x of x in M , and a T -equivariant symplectomorphism $\Phi_x: T \times_{T_x} D_x \rightarrow U_x$ with $\Phi_x([1, 0]_{T_x}) = x$. For each $t \in T$, let the mapping $\rho_x(t): D_x \rightarrow T \times_{T_x} D_x$ be given by

$$\rho_x(t)(z) = [t, z]_{T_x}. \quad (2.20)$$

Since $(\pi \circ i_x)^{-1}(\{p\}) \subset T \cdot x$, there exists a collection $\mathcal{C} := \{t_k\} \subset T$ such that $(\pi \circ i_x)^{-1}(\{p\}) = \{t_k \cdot x \mid t_k \in \mathcal{C}\}$. Since D_x is an open neighborhood of $0 \in D_x$ and Φ_x is a T -equivariant symplectomorphism, the image $\Phi_x(\text{Im}(\rho_x(t_k)))$ is an open neighborhood of $t_k \cdot x$ in \mathcal{I}_x , and the image $V_p := \pi(U_x \cap \mathcal{I}_x)$ is an open neighborhood of $p \in M/T$. Let $V(t_k \cdot x) := \Phi_x(\text{Im}(\rho_x(t_k)))$. For each $t_k \in \mathcal{C}$, the set $V(t_k \cdot x)$ is a connected subset of \mathcal{I}_x , since Φ_x and ρ_x are continuous. Each connected component K of $U_x \cap \mathcal{I}_x$ is of the form $K = V(t_k \cdot x)$ for a unique $t_k \in \mathcal{C}$. By the commutativity of diagram (2.6), the composite mapping

$$\Phi_x \circ \rho_x(t_k): D_x \rightarrow V(t_k \cdot x) \subset \mathcal{I}_x \quad (2.21)$$

is a homeomorphism and hence a chart for \mathcal{I}_x around $t_k \cdot x$, for each $t_k \in \mathcal{C}$. Since for each $t_k \in \mathcal{C}$ we have that $\pi(V(t_k \cdot x)) = V_p$, the mapping

$$\psi_x := (\pi \circ i_x) \circ (\Phi_x \circ \rho_x(1)): D_x \rightarrow V_p$$

is onto. Moreover, ψ_x is smooth and factors through the homeomorphism $\widehat{\Phi}_x: D_x/T_x \rightarrow V_p$, and hence it is an orbifold chart for M/T around p . The definitions of the maps in (2.6) imply that $\pi(V(t_k \cdot x)) = V_p$ for all $t_k \in \mathcal{C}$. The T -equivariance of Φ_x and (2.20) imply that $(\pi \circ i_x) \circ (\Phi_x \circ \rho_x(t_k)) = \psi_x$, for all $t_k \in \mathcal{C}$. By taking the neighborhood V_p around p the map $\pi \circ i_x$ is an orbifold covering as in Definition 2.20, where therein we take $U := V_p$, and the charts in (2.21) as charts for \mathcal{I}_x . \square

We will see in Section 2.4.2. that the map $\pi \circ i_x$ in Lemma 2.22 respects the symplectic forms, where the form on \mathcal{I}_x is the restriction of σ , and the form on M/T is the natural symplectic form that we define therein.

Remark 2.23 Let M be an arbitrary smooth manifold. It is a basic fact of foliation theory [59, Sec. 1] that in general, the integral manifolds of a smooth distribution D on M are injectively

immersed manifolds in M , but they are not necessarily embedded or compact. For example the one-parameter subgroup of tori $\{(t, \lambda t) + \mathbb{Z}^2 \in \mathbb{R}^2/\mathbb{Z}^2 \mid t \in \mathbb{R}\}$, in which the constant λ is an irrational real number. This is the maximal integral manifold through $(0, 0)$ of the distribution which is spanned by the constant vector field $(1, \lambda)$, and it is non-compact.

It is an exercise to verify that if I_x is a maximal integral manifold of a smooth distribution D which passes through x , then I_x must contain every end point of a smooth curve γ which starts at x and satisfies the condition that for each t its velocity vector $d\gamma(t)/dt$ belongs to $D_{\gamma(t)}$, and conversely each such end point is contained in an integral manifold through x . Therefore I_x is the set of all such endpoints, which is the unique maximal integral manifold through x . It remains to show that this set I_x is an injectively immersed manifold in M . This is one of the basic results of foliation theory, c.f. [59, Sec. 1]. In Remark 2.37 we give a self-contained proof of this fact for the particular case of our orbibundle $\pi: M \rightarrow M/T$ in Proposition 2.17. \circledcirc

Next we define a notion of orbifold fundamental group that extends the classical definition for when the orbifold is a manifold. The first difficulty is to define a “loop in an orbifold”, which we do mostly but not entirely following Boileau–Maillot–Porti [6, Sec. 2.2.1]. The definitions for orbifold loop and homotopy of loops in an orbifold which we give are not the most general ones, but give a convenient definition of orbifold fundamental group, which we use in the definition of the model for (M, σ) in Definition 2.32.

Definition 2.24 An *orbifold loop* $\alpha: [0, 1] \rightarrow \mathcal{O}$ in a smooth orbifold \mathcal{O} is represented by:

- a continuous map $\alpha: [0, 1] \rightarrow |\mathcal{O}|$ such that $\alpha(0) = \alpha(1)$ and there are at most finitely many t such that $\alpha(t)$ is a singular point of \mathcal{O} ,
- for each t such that $\alpha(t)$ is singular, a chart $\phi: \tilde{U} \rightarrow U$, $\alpha(t) \in U$, a neighborhood $V(t)$ of t in $[0, 1]$ such that for all $u \in V(t) \setminus \{t\}$, $\alpha(u)$ is regular and lies in U , and a lift $\tilde{\alpha}|_{V(t)}$ of $\alpha|_{V(t)}$ to \tilde{U} . We say that $\tilde{\alpha}|_{V(t)}$ is a *local lift* of α around t .

We say that two orbifold loops $\alpha, \alpha': [0, 1] \rightarrow \mathcal{O}$ respectively equipped with lifts of charts $\tilde{\alpha}_i, \tilde{\alpha}'_j$ represent the same loop if the underlying maps are equal and the collections of charts satisfy: for each t such that $\alpha'(t) = \alpha(t) \in U_i \cap U'_j$ is singular, where U_i, U'_j are the corresponding charts associated to t , there is a diffeomorphism ψ between a neighborhood of $\tilde{\alpha}_i(t)$ and a neighborhood of $\tilde{\alpha}'_j(t)$ such that $\psi(\tilde{\alpha}_i(s)) = \tilde{\alpha}'_j(s)$ for all s in a neighborhood of t and $\phi'_j \circ \psi = \phi_i$. \circledcirc

In Definition 2.24, if \mathcal{O} is a smooth manifold, two orbifold loops in \mathcal{O} are *equal* if their underlying maps are equal.

Definition 2.25 Let $\gamma, \lambda: [0, 1] \rightarrow \mathcal{O}$ be orbifold loops with common initial and end point in a smooth orbifold \mathcal{O} of which the set of singular points has codimension at least 2 in \mathcal{O} . We say that γ is homotopic to λ with fixed end points if there exists a continuous map $H: [0, 1] \rightarrow |\mathcal{O}|$ such that:

- for each (t, s) such that $H(t, s)$ is singular, there is a chart $\phi: \tilde{U} \rightarrow U$, $H(t, s) \in U$, a neighborhood $V(t, s)$ of (t, s) in $[0, 1]^2$ such that for all $(u, v) \in V(t, s) \setminus \{(t, s)\}$, $H(u, v)$ is regular and lies in U , and a lift $\tilde{H}|_{V(t, s)}$ of $H|_{V(t, s)}$ to \tilde{U} . We call $\tilde{H}|_{V(t, s)}$ a *local homotopy lift* of H around (t, s) .

- the orbifold loops $t \mapsto H(t, 0)$ and $t \mapsto H(t, 1)$ from $[0, 1]$ into \mathcal{O} respectively endowed with the local homotopy lifts $\tilde{H}|_{V(t, 0)}$, $\tilde{H}|_{V(t, 1)}$ of H for each $t \in [0, 1]$, are respectively equal to γ and λ as orbifold loops, c.f. Definition 2.24.
- H fixes the initial and end point: $H(0, s) = H(1, s) = \gamma(0)$ for all $s \in [0, 1]$.

∅

The assumption in Definition 2.25 on the codimension of the singular locus of the orbifold always holds for symplectic orbifolds, so such requirement is natural in our context, since the orbifold which we will be working with, the orbit space M/T , comes endowed with a symplectic structure. From now on we assume this requirement for all orbifolds.

Definition 2.26 [6, Sec. 2.2.1] Let \mathcal{O} be a connected orbifold. The *orbifold fundamental group* $\pi_1^{\text{orb}}(\mathcal{O}, x_0)$ of \mathcal{O} based at the point x_0 as the set of homotopy classes of orbifold loops with initial and end point point x_0 with the usual composition law by concatenation of loops, as in the classical sense. The set $\pi_1^{\text{orb}}(\mathcal{O}, x_0)$ is a group, and a change of base point results in an isomorphic group. ∅

It is a theorem of Thurston, c.f. [6, Th. 2.5], that any connected smooth orbifold \mathcal{O} has a unique up to equivalence orbifold covering $\tilde{\mathcal{O}}$ which is universal and whose orbifold fundamental group based at any regular point is trivial, so this definition of universal covering extends to smooth orbifolds the classical definition for smooth manifolds. Next we exhibit a construction, which is analogous to the construction of the universal cover of a generic orbifold, of such covering for the case $\mathcal{O} = M/T$.

Let $x_0 \in M$, write $p_0 = \pi(x_0) \in M/T$. By Proposition 2.12 the orbit space M/T is a compact connected and smooth $(\dim M - \dim T)$ -dimensional orbifold. For each $p \in M/T$, let $\widetilde{M/T}_p$ denote the space of homotopy classes of (orbifold) paths $\gamma: [0, 1] \rightarrow M/T$ which start at p_0 and end at p . Let $\widetilde{M/T}$ denote the set-theoretic disjoint union of the spaces $\widetilde{M/T}_p$, where p ranges over the orbit-space M/T . Let ψ be the set-theoretic mapping $\psi: \widetilde{M/T} \rightarrow M/T$ which sends $\widetilde{M/T}_p$ to p . As every smooth orbifold, the orbit space M/T has a smooth orbifold atlas $\{(U_i, \tilde{U}_i, \phi_i, \Gamma_i)\}_{i \in I}$ in which the sets \tilde{U}_i are simply connected. Pick $p_i \in U_i$, let \mathcal{F}_i be the fiber of ψ over p_i , and let x_i be a point in \mathcal{F}_i , for each $i \in I$. Let $x \in \tilde{U}_i$, $y \in \mathcal{F}_i$, choose $q_i \in \psi_i^{-1}(p_i)$ and choose a path $\tilde{\lambda}$ in \tilde{U}_i from q_i to x , and let $\lambda := \phi_i(\tilde{\lambda})$ equipped with the lift $\tilde{\lambda}$ at each point, which is an (orbifold) path in U_i such that $\lambda(0) = p_i$. By definition, the element y is a homotopy class of (orbifold) paths from p_0 to p_i . Let γ be a representative of y , and let α be the (orbifold) path obtained by concatenating γ with λ , where α travels along the points in γ first. Since \tilde{U}_i is simply connected, the homotopy class of α does not depend on the choice of γ and λ , and we define the surjective map $\psi_i(x, y) := [\alpha]$ from the Cartesian product $\tilde{U}_i \times \mathcal{F}_i$ to $\psi^{-1}(U_i)$. Equip each fiber \mathcal{F}_i with the discrete topology and the Cartesian product $\tilde{U}_i \times \mathcal{F}_i$ with the product topology. Then ψ induces a topology on $\psi^{-1}(U_i)$ whose connected components are the images of sets of the form $\tilde{U}_i \times \{y\}$ and we set $(\tilde{U}_i \times \{y\}, \psi_i|_{\tilde{U}_i \times \{y\}})$, $y \in \mathcal{F}_i$, $i \in I$, as orbifold charts for an orbifold atlas for $\widetilde{M/T}$. The pair $(\widetilde{M/T}, \psi)$ endowed with the equivalence class of atlases of the orbifold atlas for $\widetilde{M/T}$ whose orbifold charts are $(\tilde{U}_i \times \{y\}, \psi_i|_{\tilde{U}_i \times \{y\}})$, $y \in \mathcal{F}_i$, $i \in I$, is a regular and universal

orbifold covering of the orbifold M/T called the *universal cover of M/T based at p_0* , and denoted simply by $\widetilde{M/T}$, whose orbifold fundamental group $\pi_1^{\text{orb}}(\widetilde{M/T}, p_0)$ is trivial.

2.4.2 Symplectic structure on M/T

We prove that M/T comes endowed with a symplectic structure. A (*smooth*) *differential form* ω (resp. *symplectic form*) on the smooth orbifold \mathcal{O} is given by a collection $\{\tilde{\omega}_i\}$ where $\tilde{\omega}_i$ is a Γ_i -invariant differential form (resp. symplectic form) on each \tilde{U}_i and such that any two of them agree on overlaps. Here the notion of “agreeing on overlaps” can be made precise as follows: for each $x \in \tilde{U}_i$ and $y \in \tilde{U}_j$ with $\phi_i(x) = \phi_j(y)$, there is a diffeomorphism ψ between a neighborhood of x and a neighborhood of y such that $\phi_j(\psi(z)) = \phi_i(z)$ for all z in such neighborhood and $\psi^*\omega_j = \omega_i$. A *symplectic orbifold* is a smooth orbifold equipped with a symplectic form.

Remark 2.27 Let ω be a differential form on an orbifold \mathcal{O} , and suppose that ω is given by a collection $\{\tilde{\omega}_i\}$ where $\tilde{\omega}_i$ is a Γ_i -invariant differential form (resp. symplectic form) on each \tilde{U}_i . Because the $\tilde{\omega}_i$'s which define it are Γ_i -invariant and agree on overlaps, ω is uniquely determined by its values on any orbifold atlas for \mathcal{O} even if the atlas is not maximal. \circlearrowright

If $f: \mathcal{O}' \rightarrow \mathcal{O}$ is an orbifold diffeomorphism, the *pull-back of a differential form* ω on \mathcal{O} is the unique differential form ω' on \mathcal{O}' given by $\tilde{f}^*\tilde{\omega}_i$ on each chart $\tilde{f}^{-1}(\tilde{U}_i)$; we write $\omega' := f^*\omega$. Analogously we define the pullback under a principal T -orbibundle $p: Y \rightarrow X/T$ as in Definition 2.15, of a form ω on X/T , where the maps \tilde{p} are given in Remark 2.16.

Lemma 2.28. *Let (M, σ) be a compact, connected, symplectic manifold equipped with an effective (resp. free) symplectic action of a torus T for which at least one, and hence every T -orbit is a symplectic $\dim T$ -dimensional submanifold of (M, σ) . Then there exists a unique 2-form ν on the orbit space M/T such that $\pi^*\nu|_{\Omega_x} = \sigma|_{\Omega_x}$ for every $x \in M$, where $\{\Omega_x\}_{x \in M}$ is the distribution on M of symplectic orthogonal complements to the tangent spaces to the T -orbits, and $\pi: M \rightarrow M/T$ is the projection map, c.f. Proposition 2.17. Moreover, the form ν is symplectic, and so the pair $(M/T, \nu)$ is a compact, connected symplectic orbifold (resp. manifold).*

Proof. By Corollary 2.19, for each $x \in M$ there is a T -invariant neighborhood U_x of the T -orbit $T \cdot x$ and a T -equivariant symplectomorphism Φ_x from U_x onto $T \times_{T_x} D_x$ with the symplectic form $\widetilde{\sigma^T \oplus \sigma^{\mathbb{C}^m}}$. As in Proposition 2.12, the collection of charts $\{(\pi(U_x), D_x, \phi_x, T_x)\}$ given by expression (in the general definition of smooth orbifold $\tilde{U}_i = D_x$, $U_i = \pi(U_x)$) (2.9) is an orbifold atlas for the orbit space M/T , which is equivalent to the one given in Proposition 2.12 and hence defines the orbifold structure of M/T given in Definition 2.13. Because by Remark 2.27 it suffices to define a smooth differential form on the charts of any atlas of our choice, the collection $\{\nu_x\}_{x \in M}$ given by $\nu_x := \sigma^{\mathbb{C}^m}$ defines a unique smooth differential 2-form on the orbit space M/T . Because $\sigma^{\mathbb{C}^m}$ is moreover symplectic, each ν_x is a symplectic form, and hence so is ν on M/T . \square

We say that two symplectic orbifolds (\mathcal{O}_1, ν_1) , (\mathcal{O}_2, ν_2) are *symplectomorphic* if there is an orbifold diffeomorphism $f: \mathcal{O}_1 \rightarrow \mathcal{O}_2$ with $f^*\nu_2 = \nu_1$. f is called an *orbifold symplectomorphism*.

Lemma 2.29. *Let $(M, \sigma), (M', \sigma')$ be compact connected symplectic manifolds equipped with an effective (resp. free) symplectic action of a torus T for which at least one, and hence every T -orbit is a $\dim T$ -dimensional symplectic submanifold of (M, σ) and (M', σ') , respectively. Suppose additionally that (M, σ) is T -equivariantly symplectomorphic to (M', σ') . Then the symplectic orbit spaces $(M/T, \nu)$ and $(M'/T, \nu')$ are symplectomorphic.*

Proof. By Lemma 2.4, every T -orbit is a $\dim T$ -dimensional symplectic submanifold of (M', σ') . Indeed, assume that Φ is a T -equivariant symplectomorphism from the symplectic manifold (M, σ) to the symplectic manifold (M', σ') . Because the mapping Φ is T -equivariant, it descends to an orbifold diffeomorphism $\widehat{\Phi}$ from the orbit space M/T onto the orbit space M'/T . Because of the definition of the connections Ω and of Ω' for the orbibundle projections $\pi: M \rightarrow M'/T$ and $\pi': M' \rightarrow M'/T$ respectively, as the connections given by the symplectic orthogonal complements to the tangent spaces to the T -orbits, we have that $\Phi^*\Omega' = \Omega$.

Since by Lemma 2.28 the symplectic forms ν on M/T and ν' on M'/T are respectively induced by the projections π and π' , and both $\Phi^*\Omega' = \Omega$ and $\Phi^*\sigma' = \sigma$, we have that $\widehat{\Phi}^*\nu' = \nu$ and therefore $\widehat{\Phi}$ is an orbifold symplectomorphism between $(M/T, \nu)$ and $(M'/T, \nu')$. \square

We use the notation $\int_{\mathcal{O}} \omega$ for the integral of the differential form ω on the orbifold \mathcal{O} . If (\mathcal{O}, ω) is a symplectic orbifold, the integral $\int_{\mathcal{O}} \omega$ is known as the *total symplectic area of (\mathcal{O}, ω)* .

Remark 2.30 Under the assumptions of Lemma 2.29, if $\dim T = \dim M - 2$, then the total symplectic area of the symplectic orbit space $(M/T, \nu)$ equals the total symplectic area of $(M'/T, \nu')$. Moreover, recall that by the orbifold Moser's theorem [38, Th. 3.3], if $(M, \sigma), (M', \sigma')$ are any symplectic manifolds equipped with an action of a torus T of dimension $\dim T = \dim M - 2$, for which at least one and hence every T -orbit is a $\dim T$ -dimensional symplectic submanifold of (M, σ) , and such that the symplectic area of the orbit space $(M/T, \nu)$ equals the symplectic area of the orbit space $(M'/T, \nu')$, and the Fuchsian signature of the orbisurfaces M/T and M'/T are equal, then $(M/T, \nu)$ is T -equivariantly (orbifold) symplectomorphic to $(M'/T, \nu')$; we emphasize that this is the case because the orbit spaces M/T and M'/T are 2-dimensional, so their genus determines their diffeomorphism type. This result is used in the proof of Theorem 3.15 and Theorem 4.33. \oslash

2.4.3 Model of (M, σ) : definition

In the following definition we define a T -equivariant symplectic model for (M, σ) : for each regular point $p_0 \in M/T$ we construct a smooth manifold M_{model, p_0} , a T -invariant symplectic form σ_{model} on M_{model, p_0} , and an effective symplectic action of the torus T on M_{model, p_0} . The ingredients for the construction of such model are Proposition 2.17 and Lemma 2.28.

If $p: \mathcal{O}' \rightarrow \mathcal{O}$ is an orbifold covering map, p is a local diffeomorphism in the sense of Remark 2.21, and the *pull-back of a differential form ω on \mathcal{O}* is the unique differential form ω' on \mathcal{O}' given by $\tilde{p}^*\tilde{\omega}_i$ on each chart $\tilde{p}^{-1}(\tilde{U}_i)$; we write $\omega' := p^*\omega$.

Remark 2.31 This remark justifies that item ii) in Definition 2.32 below is correctly defined. The reader may proceed with Definition 2.32 and return to this remark as therein indicated.

The pull-back of the symplectic form ν on M/T , given by Lemma 2.28, to the universal cover $\widetilde{M/T}$, by means of the smooth covering map $\psi: \widetilde{M/T} \rightarrow M/T$, is a $\pi_1^{\text{orb}}(M/T, p_0)$ -invariant symplectic form on the orbifold universal cover $\widetilde{M/T}$. The symplectic form on the torus T is translation invariant and therefore $\pi_1^{\text{orb}}(M/T, p_0)$ -invariant. The direct sum of the symplectic form on $\widetilde{M/T}$ and the symplectic form on T is a $\pi_1^{\text{orb}}(M/T, p_0)$ -invariant (and T -invariant) symplectic form on the Cartesian product $\widetilde{M/T} \times T$, and therefore there exists a unique symplectic form on the associated bundle $\widetilde{M/T} \times_{\pi_1^{\text{orb}}(M/T, p_0)} T$ of which the pull-back by the covering map $\widetilde{M/T} \times T \rightarrow \widetilde{M/T} \times_{\pi_1^{\text{orb}}(M/T, p_0)} T$ is equal to the given symplectic form on $\widetilde{M/T} \times T$. \oslash

Definition 2.32 Let (M, σ) be a compact, connected, symplectic manifold equipped with an effective symplectic action of a torus T for which at least one, and hence every T -orbit is a $\dim T$ -dimensional symplectic submanifold of (M, σ) . We define the *T -equivariant symplectic model* $(M_{\text{model}, p_0}, \sigma_{\text{model}})$ of (M, σ) based at a regular point $p_0 \in M/T$ as follows.

- i) The space M_{model, p_0} is the associated bundle

$$M_{\text{model}, p_0} := \widetilde{M/T} \times_{\pi_1^{\text{orb}}(M/T, p_0)} T,$$

where the space $\widetilde{M/T}$ denotes the orbifold universal cover of the orbifold M/T based at a regular point $p_0 \in M/T$, and the orbifold fundamental group $\pi_1^{\text{orb}}(M/T, p_0)$ acts on the product $\widetilde{M/T} \times T$ by the diagonal action $x(y, t) = (x \star y^{-1}, \mu(x) \cdot t)$, where $\star: \pi_1^{\text{orb}}(M/T, p_0) \times \widetilde{M/T} \rightarrow \widetilde{M/T}$ denotes the natural action of $\pi_1^{\text{orb}}(M/T, p_0)$ on $\widetilde{M/T}$, and $\mu: \pi_1^{\text{orb}}(M/T, p_0) \rightarrow T$ denotes the monodromy homomorphism of the flat connection $\Omega := \{\Omega_x\}_{x \in M}$ given by the symplectic orthogonal complements to the tangent spaces to the T -orbits (c.f. Proposition 2.17).

- ii) The symplectic form σ_{model} is induced on the quotient by the product symplectic form on the Cartesian product $\widetilde{M/T} \times T$. The symplectic form on $\widetilde{M/T}$ is defined as the pullback by the orbifold universal covering map $\widetilde{M/T} \rightarrow M/T$ of the unique 2-form ν on M/T such that $\pi^*\nu|_{\Omega_x} = \sigma|_{\Omega_x}$ for every $x \in M$ (c.f. Lemma 2.28). The symplectic form on the torus T is the unique T -invariant symplectic form determined by the non-degenerate antisymmetric bilinear form σ^t such that $\sigma_x(X_M(x), Y_M(x)) = \sigma^t(X, Y)$, for every $X, Y \in \mathfrak{t}$, and every $x \in M$ (c.f. Lemma 2.1). See Remark 2.31.
- iii) The action of T on the space M_{model, p_0} is the action of T by translations which descends from the action of T by translations on the right factor of the product $\widetilde{M/T} \times T$.

\oslash

Example 2.33 When $\dim M - \dim T = 2$ the orbit space M/T is 2-dimensional and it is an exercise to describe locally the monodromy homomorphism μ which appears in part i) of Definition 2.32. A small T -invariant open subset of our symplectic manifold looks like $T \times_{T_x} D$, where T

is the standard 2-dimensional torus $(\mathbb{R}/\mathbb{Z})^2$, and D is a standard 2-dimensional disk centered at the origin in the complex plane \mathbb{C} . Here the quotient M/T is the orbisurface D/T_x . (Recall that we know that T_x is a finite cyclic group, c.f. Lemma 2.6). Suppose that T_x has order n . The monodromy homomorphism μ in Definition 2.32 part i) is a map from $\pi_1^{\text{orb}} := \pi_1^{\text{orb}}(D/T_x, p_0)$ into T . Note that $\pi_1^{\text{orb}} = \langle \gamma \rangle$ with γ of order n . If $t \in T$, there exists a homomorphism $f: \pi_1^{\text{orb}} \rightarrow T$ such that $f(\gamma) = t$ if and only if $t^n = 1$.

If we identify T with $(\mathbb{R}/\mathbb{Z})^2$, we can write $t = (t_1, t_2)$, there exists a homomorphism $f: \pi_1^{\text{orb}} \rightarrow T$ such that $f(\gamma) = (t_1, t_2)$ if and only if n divides the order of (t_1, t_2) , which means that t_1, t_2 must be rational numbers such that n divides the smallest integer m such that $m t_i \in \mathbb{Z}$, $i = 1, 2$. If for example $n = 2$, this condition says that the smallest integer m such that $m t_i \in \mathbb{Z}$ must be an even number. In other words, not every element in T can be achieved by the monodromy homomorphism. In fact, all the achievable elements are of finite order, but as we see from this example, more restrictions must take place. \oslash

2.4.4 Model of (M, σ) : proof

We prove that the associated bundle in Definition 2.32, which we called “the model of M ”, is T -equivariantly symplectomorphic to (M, σ) . The main ingredient of the proof is the existence of the flat connection for $\pi: M \rightarrow M/T$ in Proposition 2.17.

We start with the observation that the universal cover $\widetilde{M/T}$ is a smooth manifold and the orbit space M/T is a good orbifold. Recall that an orbifold \mathcal{O} is said to be *good* (resp. *very good*) if it is obtained as the quotient of manifold by a discrete (resp. finite) group of diffeomorphisms of such manifold.

Lemma 2.34. *Let (M, σ) be a compact, connected, symplectic manifold equipped with an effective symplectic action of a torus T for which at least one, and hence every T -orbit is a $\dim T$ -dimensional symplectic submanifold of (M, σ) . Then the orbifold universal cover $\widetilde{M/T}$ is a smooth manifold and the orbit space M/T is a good orbifold. Moreover, if $\dim T = \dim M - 2$, the orbit space M/T is a very good orbifold.*

Proof. Let $\psi: \widetilde{M/T} \rightarrow M/T$ be the universal cover of orbit space M/T based at the regular point $p_0 = \pi(x_0) \in M/T$. Recall from Proposition 2.17 the connection Ω for the principal T -orbibundle $\pi: M \rightarrow M/T$ projection mapping, whose elements are the symplectic orthogonals to the tangent spaces to the T -orbits of (M, σ) . By Corollary 2.22, if the mapping $i_x: \mathcal{I}_x \rightarrow M$ is the inclusion mapping of the integral manifold \mathcal{I}_x through x to the distribution Ω , then the composite map $\pi \circ i_x: \mathcal{I}_x \rightarrow M/T$ is an orbifold covering mapping for each $x \in M$, in which the total space \mathcal{I}_x is a smooth manifold. Because the covering map $\psi: \widetilde{M/T} \rightarrow M/T$ is universal, there exists an orbifold covering $r: \widetilde{M/T} \rightarrow \mathcal{I}_x$ such that $\pi \circ i_x \circ r = \psi$, and in particular the orbifold universal cover $\widetilde{M/T}$ is a smooth manifold, since an orbifold covering of a smooth manifold must be a smooth manifold itself. Since the orbit M/T is obtained as a quotient of $\widetilde{M/T}$ by the discrete group $\pi_1^{\text{orb}}(M/T, p_0)$, by definition $(M/T, \mathcal{A})$ is a good orbifold. Now, it is well-known [6, Sec. 2.1.2] that in dimension 2, an orbifold is good if and only if it is very good. \square

Let $\alpha: [0, 1] \rightarrow \widehat{\mathcal{O}}$ be an orbifold path in $\widehat{\mathcal{O}}$. The *projection of the path α under a covering mapping $p: \widehat{\mathcal{O}} \rightarrow \mathcal{O}$* , which we write as $p(\alpha)$, is a path in the orbifold \mathcal{O} whose underlying map is $p \circ \alpha$, and such that for each t for which $(p \circ \alpha)(t)$ is a singular point, if $\alpha(t)$ is regular then there exists a neighborhood V of $\alpha(t)$ such that $p|_V$ is a chart at $(p \circ \alpha)(t)$, and this can be used to define the local lift of $p \circ \alpha$ around t ; if otherwise $\alpha(t)$ is singular, then by definition of α there is a chart $\phi: \tilde{V} \rightarrow V$ at $\alpha(t)$ and a local lift to \tilde{V} of α around t , and by choosing \tilde{V} to be small enough, $\phi \circ p$ is a chart of $(p \circ \alpha)(t)$ giving the local lift of $p \circ \alpha$ around t . We say that α is a *lift of a path β in \mathcal{O}* if $p(\alpha) = \beta$. The following generalizes the classical manifold property. Recall Definition 2.15.

Lemma 2.35. *Let T be a torus, let (X, ω) be a compact, connected symplectic manifold endowed with an effective symplectic action of T for which at least one, and hence every T -orbit is a $\dim T$ -dimensional symplectic submanifold of (X, ω) . Equip X/T with the orbifold structure in Definition 2.13. Let $p_0 \in X/T$ be a regular point. Let H be a connection for a smooth principal T -orbibundle $p: X \rightarrow X/T$. Then for any loop $\gamma: [0, 1] \rightarrow X/T$ in the orbifold X/T such that $\gamma(0) = p_0$ there exists a unique horizontal lift $\lambda_\gamma: [0, 1] \rightarrow X$ with respect to the connection H for $p: X \rightarrow X/T$, such that $\lambda_\gamma(0) = x_0$, where by horizontal we mean that $d\lambda_\gamma(t)/dt \in H_{\lambda_\gamma(t)}$ for every $t \in [0, 1]$.*

The following is the main result of Section 2.

Theorem 2.36. *Let (M, σ) be a compact, connected, symplectic manifold equipped with an effective symplectic action of a torus T , for which at least one, and hence every T -orbit is a $\dim T$ -dimensional symplectic submanifold of (M, σ) . Then (M, σ) is T -equivariantly symplectomorphic to its T -equivariant symplectic model based at a regular point $p_0 \in M/T$, c.f. Definition 2.32, for every regular point $p_0 \in M/T$.*

Proof. Let $\psi: \widetilde{M/T} \rightarrow M/T$ be the universal cover of orbit space M/T based at the point $p_0 = \pi(x_0) \in M/T$. Recall from Proposition 2.17 the connection Ω for the T -orbibundle $\pi: M \rightarrow M/T$ projection mapping given by the symplectic orthogonals to the tangent spaces to the T -orbits of (M, σ) . By Lemma 2.34, $\widetilde{M/T}$ is a smooth manifold. The mapping $\pi_1^{\text{orb}}(M/T, p_0) \times \widetilde{M/T} \rightarrow \widetilde{M/T}$, $([\lambda], [\gamma]) \mapsto [\gamma \lambda]$, is a smooth action of the orbifold fundamental group $\pi_1^{\text{orb}}(M/T, p_0)$ on the orbifold universal cover $\widetilde{M/T}$, which is transitive on each fiber $\widetilde{M/T}_p$ of $\psi: \widetilde{M/T} \rightarrow M/T$. By Lemma 2.35, for any loop $\gamma: [0, 1] \rightarrow M/T$ in the orbifold M/T such that $\gamma(0) = p_0$, denote by $\lambda_\gamma: [0, 1] \rightarrow M$ its unique horizontal lift with respect to the connection Ω for $\pi: M \rightarrow M/T$ such that $\lambda_\gamma(0) = x_0$, where by horizontal we mean that $d\lambda_\gamma(t)/dt \in \Omega_{\lambda_\gamma(t)}$ for every $t \in [0, 1]$. Proposition 2.17 says that Ω is an orbifold flat connection, which means that $\lambda_\gamma(1) = \lambda_\delta(1)$ if δ is homotopy equivalent to γ in the space of all orbifold paths in the orbit space M/T which start at p_0 and end at the given end point $p = \gamma(1)$. This gives existence of a unique group homomorphism $\mu: \pi_1^{\text{orb}}(M/T, p_0) \rightarrow T$ such that $\lambda_\gamma(1) = \mu([\gamma]) \cdot x_0$. The homomorphism μ does not depend on the choice of the base point $x_0 \in M$. For any homotopy class $[\gamma] \in \widetilde{M/T}$ and $t \in T$, define the element

$$\Phi([\gamma], t) := t \cdot \lambda_\gamma(1) \in M. \quad (2.22)$$

The assignment $([\gamma], t) \mapsto \Phi([\gamma], t)$ defines a smooth covering mapping $\Phi: \widetilde{M/T} \times T \rightarrow M$ between smooth manifolds. Let $[\delta] \in \pi_1^{\text{orb}}(M/T, p_0)$ act on the product $\widetilde{M/T} \times T$ by sending the pair $([\gamma], t)$ to the pair $([\gamma \delta^{-1}], \mu([\delta])t)$. The mapping Φ induces a diffeomorphism ϕ from the associated bundle $\widetilde{M/T} \times_{\pi_1^{\text{orb}}(M/T, p_0)} T$ onto M . Indeed, ϕ is onto because Φ is onto, and $\Phi([\gamma], t) = \Phi([\gamma'], t')$ if and only if $t \cdot \lambda_\gamma(1) = t' \cdot \lambda_{\gamma'}(1)$, if and only if $\lambda_\gamma(1) = (t^{-1}t') \cdot \lambda_{\gamma'}(1)$, if and only if $t^{-1}t' = \mu([\delta])$, in which δ is equal to the loop starting and ending at p_0 , which is obtained by first doing the path γ and then going back by means of the path γ'^{-1} . Since Φ is a smooth covering map, the mapping ϕ is a local diffeomorphism, and we have just proved that it is bijective, so ϕ must be a diffeomorphism. By definition, ϕ intertwines the action of T by translations on the right factor of the associated bundle $\widetilde{M/T} \times_{\pi_1^{\text{orb}}(M/T, p_0)} T$ with the action of T on M . Recall that the symplectic form on $\widetilde{M/T} \times_{\pi_1^{\text{orb}}(M/T, p_0)} T$ is the unique symplectic form of which the pullback $\widetilde{M/T} \times T \rightarrow \widetilde{M/T} \times_{\pi_1^{\text{orb}}(M/T, p_0)} T$ is equal to the product form on $\widetilde{M/T} \times T$, where the symplectic form on $\widetilde{M/T}$ is given by Definition 2.32 part ii). It follows from the definition of the symplectic form on $\widetilde{M/T} \times_{\pi_1^{\text{orb}}(M/T, p_0)} T$ and Corollary 2.19 that the T -equivariant diffeomorphism ϕ from $\widetilde{M/T} \times_{\pi_1^{\text{orb}}(M/T, p_0)} T$ onto M pulls back the symplectic form σ on M to the just obtained symplectic form on $\widetilde{M/T} \times_{\pi_1^{\text{orb}}(M/T, p_0)} T$. \square

Remark 2.37 We assume the terminology of Definition 2.32 and Theorem 2.36. In this remark we describe a covering isomorphic to the orbifold covering $\pi \circ i_x$ of Lemma 2.22, which will be of use in Theorem 2.47. Additionally, the construction of this new covering makes transparent the relation between the universal cover $\widetilde{M/T} \rightarrow M/T$ and the covering $\mathcal{I}_x \rightarrow M/T$. Indeed, recall the distribution Ω of the symplectic orthogonal complements of the tangent spaces of the T -orbits given in Proposition 2.17, and the principal T -orbibundle $\pi: M \rightarrow M/T$, for which Ω is a T -invariant flat connection. As in the proof of Theorem 2.36, if $x \in M$ then each smooth curve δ in M/T which starts at $\pi(x)$ has a unique horizontal lift γ which starts at x . The endpoints of such lifts form the injectively immersed manifold \mathcal{I}_x , c.f. Remark 2.23. As in the proof of Theorem 2.36, if we keep the endpoints of δ fixed, then the endpoint of γ only depends on the homotopy class of δ . As explained prior to Definition 2.32, the homotopy classes of δ 's in M/T with fixed endpoints are by definition the elements of the universal covering space $\widetilde{M/T}$ of M/T , which is a principal $\pi_1^{\text{orb}}(M/T, \pi(x))$ -bundle over M/T . The corresponding endpoints of the γ 's exhibit the integral manifold \mathcal{I}_x as the image of an immersion from $\widetilde{M/T}$ into M , but this immersion is not necessarily injective. Recall that the monodromy homomorphism μ of Ω tells what the endpoint of γ is when δ is a loop. By replacing μ by the induced injective homomorphism μ' from $\pi_1^{\text{orb}}(M/T, \pi(x))/\ker \mu$ to T , we get an injective immersion. This procedure is equivalent to replacing the universal covering $\widetilde{M/T}$ by the covering $\widetilde{M/T}/\ker \mu$ with fiber $\pi_1^{\text{orb}}(M/T, \pi(x))/\ker \mu$. So $\widetilde{M/T}/\ker \mu$ is injectively immersed in M by a map whose image is \mathcal{I}_x . It follows that the coverings $\pi \circ i_x: \mathcal{I}_x \rightarrow M/T$ and $\widetilde{M/T}/\ker \mu \rightarrow M/T$ are isomorphic coverings of M/T , of which the total space is a smooth manifold. We use this new covering $\widetilde{M/T}/\ker \mu \rightarrow M/T$ in order to construct an alternative model of (M, σ) to the one given in Theorem 2.36, c.f. Theorem 2.47. Note: in the statement of Theorem

2.47, $N := \widetilde{M/T}/\ker \mu$.

∅

Example 2.38 Assume the terminology of Definition 2.32. Assume moreover that $\dim M - \dim T = 2$ and that the action of T on M is free, so the quotient space M/T is a compact, connected, smooth surface and it is classified by its genus. If the genus is zero, then the orbit space M/T is a sphere, which is simply connected, and the orbibundle $\pi : \mathcal{I}_x \rightarrow M/T$ is a diffeomorphism, c.f. Lemma 2.22. In such case M is the Cartesian product of a sphere with a torus. If M/T has genus 1, then the orbit space M/T is a two-dimensional torus, with fundamental group at any point isomorphic to the free abelian group on two generators t_1, t_2 , and the monodromy homomorphism is determined by the images t_1 and t_2 of the generators $(1, 0)$ and $(0, 1)$ of \mathbb{Z}^2 . \mathcal{I}_x is compact if and only if \mathcal{I}_x is a closed subset of M if and only if t_1 and t_2 generate a closed subgroup of T if and only if t_1 and t_2 generate a finite subgroup of T . \mathcal{I}_x is dense in M if and only if the subgroup of T generated by t_1 and t_2 is dense in T . If the genus of M/T is strictly positive, then \mathcal{I}_x is compact if and only if the monodromy elements form a finite subgroup of T , which is a very particular situation (since the $2g$ generators of the monodromy subgroup of T can be chosen arbitrarily, this is very rare: even one element of T usually generates a dense subgroup of T). ∅

2.5 Model up to T -equivariant diffeomorphisms

2.5.1 Generalization of Kahn's theorem

In [25, Cor. 1.4] P. Kahn states that if a compact connected 4-manifold M admits a free action of a 2-torus T such that the T -orbits are 2-dimensional symplectic submanifolds, then M splits as a product $M/T \times T$, the following being the statement in [25].

Theorem 2.39 (P. Kahn, [25], Cor. 1.4). *Let (M, σ) be a compact, connected symplectic 4-dimensional manifold. Suppose that M admits a free action of a 2-dimensional torus T for which the T -orbits are 2-dimensional symplectic submanifolds of (M, σ) . Then there exists a T -equivariant diffeomorphism between M and the cartesian product $M/T \times T$, where T is acting by translations on the right factor of $M/T \times T$.*

Next we generalize this result of Kahn's to the case when the torus and the manifold are of arbitrary dimension, c.f Theorem 2.41. We start with Corollary 2.40, the proof of which relies on the forthcoming Theorem 2.41, so we suggest to return to its proof at the end of the section.

Corollary 2.40. *Let (M, σ) be a compact connected symplectic $2n$ -dimensional manifold equipped with a free symplectic action of a $(2n - 2)$ -dimensional torus T such that at least one, and hence every T -orbit is a $(2n - 2)$ -dimensional symplectic submanifold of (M, σ) . Then M is T -equivariantly diffeomorphic to the Cartesian product $M/T \times T$, where $M/T \times T$ is equipped with the action of T on the right factor of $M/T \times T$ by translations.*

Proof. If $\dim(M/T) = 2$, by the first part of Theorem 2.41, the quotient space M/T is a compact, connected, smooth, orientable surface. Therefore, by the classification theorem for surfaces, $H_1(M/T, \mathbb{Z})$ is isomorphic to \mathbb{Z}^{2g} , where g is the topological genus of M/T . Therefore

$H_1(M/T, \mathbb{Z})$ does not have torsion elements, and by Theorem 2.41, M is T -equivariantly diffeomorphic to the cartesian product $M/T \times T$. \square

2.5.2 Smooth equivariant splittings

We give a characterization of the existence of T -equivariant splittings of M as a Cartesian product $M/T \times T$, up to T -equivariant diffeomorphisms c.f. Theorem 2.41.

Recall that if X is a smooth manifold, $H_1(X, \mathbb{Z})_T$ denotes the subgroup of torsion elements of $H_1(X, \mathbb{Z})$ i.e. the set of $[\gamma] \in H_1(X, \mathbb{Z})$ such that there exists a strictly positive integer k satisfying $k[\gamma] = 0$. $H_1(X, \mathbb{Z})_T$ is a subgroup. Let μ_h^T denote the homomorphism given by restricting the homomorphism μ_h to $H_1(M/T, \mathbb{Z})_T$. Recall that by Proposition 2.12, M is a compact, connected $(\dim M - \dim T)$ -dimensional orbifold, and that if the action of torus T on M is free, the local groups are trivial, and hence the orbit space M/T is a $(\dim M - \dim T)$ -dimensional smooth manifold, c.f. Remark 2.14.

Theorem 2.41. *Let (M, σ) be a compact, connected, symplectic manifold equipped with a free symplectic action of a torus T for which at least one, and hence every of its T -orbits is a $\dim T$ -dimensional symplectic submanifold of (M, σ) , and let $\mu_h: H_1(M/T, \mathbb{Z}) \rightarrow T$ be the homomorphism induced on homology via the Hurewicz map by the monodromy homomorphism μ from $\pi_1(M/T, p_0)$ into T with respect to the connection on M given by the symplectic orthogonal complements to the tangent spaces to the T -orbits, for the T -bundle projection map $\pi: M \rightarrow M/T$ onto the orbit space M/T (c.f. expression (2.25)). Then M is T -equivariantly diffeomorphic to the Cartesian product $M/T \times T$ equipped with the action of T by translations on the right factor of $M/T \times T$, if and only if the torsion part μ_h^T of the homomorphism μ_h is trivial, i.e. μ_h satisfies that $\mu_h([\gamma]) = 1$ for every $[\gamma] \in H_1(M/T, \mathbb{Z})$ of finite order.*

Proof. By means of the same argument that we used in the proof of Theorem 2.36, using any T -invariant flat connection \mathcal{D} for π instead of Ω , we may define a mapping

$$\phi^{\mathcal{D}}: \widetilde{M/T} \times_{\pi_1(M/T, p_0)} T \rightarrow M. \quad (2.23)$$

The mapping $\phi^{\mathcal{D}}$, which is induced by (2.22), is a T -equivariant diffeomorphism between the smooth manifolds $\widetilde{M/T} \times_{\pi_1(M/T, p_0)} T$ and M , where $\widetilde{M/T}$ is the universal cover of the manifold M/T based at p_0 . Since $\widetilde{M/T}$ is a regular covering c.f. Remark 2.43, $\widetilde{M/T}/\pi_1(M/T, p_0) = M/T$ and the monodromy homomorphism mapping $\mu^{\mathcal{D}}: \pi_1(M/T, p_0) \rightarrow T$ of \mathcal{D} is trivial if and only if $\widetilde{M/T} \times_{\pi_1(M/T, p_0)} T = M/T \times T$. Hence the smooth manifold M is T -equivariantly diffeomorphic to the Cartesian product $M/T \times T$ if and only if there exists a T -invariant flat connection \mathcal{D} for the principal T -bundle $\pi: M \rightarrow M/T$ for which its monodromy homomorphism $\mu^{\mathcal{D}}: \pi_1(M/T, p_0) \rightarrow T$ is trivial.

If $\mu_h^T = 0$, there exists a homomorphism $\widehat{\mu}_h: H_1(M/T, \mathbb{Z}) \rightarrow \mathfrak{t}$ such that $\exp \circ \widehat{\mu}_h = \mu_h$, which by viewing $\widehat{\mu}_h$ as an element $[\beta] \in H_{\text{de Rham}}^1(M/T) \otimes \mathfrak{t}$, can be rewritten as

$$\mu_h([\gamma]) = \exp \int_{\gamma} \beta. \quad (2.24)$$

The T -invariant connection Ω of symplectic orthogonal complements to the tangent spaces to the T -orbits defines a T -invariant connection one-form θ for the canonical projection mapping $\pi: M \rightarrow M/T$, which, by T -invariance of θ , is identified with a connection one-form $\widehat{\theta} \in \Omega^1(M/T) \otimes \mathfrak{t}$. By Proposition 2.17, the connection Ω is flat, and since the torus T is an abelian group, $\widehat{\theta}$ is a closed form. The one-form $\Theta := \widehat{\theta} - \beta \in \Omega_{\text{de Rham}}^1(M/T) \otimes \mathfrak{t}$ is a \mathfrak{t} -valued connection one-form on M/T , in the sense that its pull-back to M is a connection one-form on M , and therefore there exists a T -invariant connection \mathcal{D} on M , such that $\Theta = \widehat{\theta}^{\mathcal{D}}$. Since β is closed, and Ω is flat, the connection one-form $\widehat{\theta}^{\mathcal{D}}$ is flat, and hence the connection \mathcal{D} on M is flat. This means that \mathcal{D} has a monodromy homomorphism $\mu^{\mathcal{D}}: \pi_1(M/T, p_0) \rightarrow T$, and a corresponding monodromy homomorphism on homology $\mu_h^{\mathcal{D}}: H_1(M/T, \mathbb{Z}) \rightarrow \mathfrak{t}$, and for any closed curve γ in the orbit space M/T ,

$$\mu_h^{\mathcal{D}}([\gamma]) = \mu_h([\gamma]) \exp \int_{\gamma} -\beta = \mu_h([\gamma]) (\exp \int_{\gamma} \beta)^{-1} = 1,$$

where the second equality follows from the fact that

$$\exp \int_{\gamma} -\beta = (\exp \int_{\gamma} \beta)^{-1},$$

and the third equality follows from (2.24), and we have proven that \mathcal{D} is a flat connection for the orbibundle $\pi: M \rightarrow M/T$ whose monodromy homomorphism on homology $\mu_h^{\mathcal{D}}$ is trivial, and hence so it is $\mu^{\mathcal{D}}$. If conversely, M is T -equivariantly diffeomorphic to the Cartesian product $M/T \times T$, then there exists a T -invariant flat connection \mathcal{D} for the orbibundle $\pi: M \rightarrow M/T$ such that the monodromy homomorphism for π with respect to \mathcal{D} , $\mu^{\mathcal{D}}: \pi_1(M/T, p_0) \rightarrow T$ is trivial. On the other hand, if $[\gamma] \in H_1(M/T, \mathbb{Z})_T$, there exists a strictly positive integer k with $k [\gamma] = 0$ and $0 = \langle k [\gamma], [\alpha] \rangle = k \langle [\gamma], [\alpha] \rangle$, which means that $\langle [\gamma], [\alpha] \rangle = 0$ for all $[\gamma] \in H_1(M/T, \mathbb{Z})_T$, and hence that $(\mu_h^{\mathcal{D}})^T = \mu_h^T$. Since the homomorphism $\mu^{\mathcal{D}}$ is trivial, μ_h^T is trivial. \square

Remark 2.42 Under the assumptions of Theorem 2.41 on our symplectic manifold (M, σ) , if the dimension of the orbit space M/T is strictly greater than 2, then it may happen that the first integral orbifold homology group $H_1(M/T, \mathbb{Z})$ has no torsion and therefore M is T -equivariantly diffeomorphic to the Cartesian product $M/T \times T$. However, already in the case that M/T is 4-dimensional there are examples of symplectic manifolds M/T of which the the integral homology group $H_1(M/T, \mathbb{Z})$ has nontrivial torsion. For instance, in [12, Sec. 8] we computed the fundamental group and the first integral homology group of the whole manifold, and this computation shows that even in dimension 4 there are examples X where $H_1(X, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^3 \times (\mathbb{Z}/k\mathbb{Z})$, where k can be any positive integer. Taking such manifolds as the base space M/T , and a monodromy homomorphism which is nontrivial on the torsion subgroup $\mathbb{Z}/k\mathbb{Z}$, we arrive at an example where M is not T -equivariantly diffeomorphic to the Cartesian product $M/T \times T$. \circlearrowright

Remark 2.43 Under the assumptions of Theorem 2.41, since $\widetilde{M/T}$ is a regular covering of M/T , we have a diffeomorphism $\widetilde{M/T}/\pi_1(M/T, p_0) \simeq M/T$ naturally, or in other words, the symbol \simeq may be taken to be an equality. Hence the monodromy homomorphism $\mu^{\mathcal{D}}: \pi_1(M/T, p_0) \rightarrow T$

of a T -invariant flat connection \mathcal{D} is trivial if and only if $\widetilde{M/T} \times_{\pi_1(M/T, p_0)} T = M/T \times T$. Strictly speaking, this is not an equality, it is a T -equivariant diffeomorphism

$$\widetilde{M/T} \times_{\pi_1(M/T, p_0)} T \rightarrow \widetilde{M/T}/\pi_1(M/T, p_0) \times T \rightarrow M/T \times T,$$

where the first arrow is the identity map, and the second arrow is the identity on the T component of the Cartesian product $(\widetilde{M/T}/\pi_1(M/T, p_0)) \times T$, while on the first component it is given by the map $[\gamma]_{\pi_1(M/T, p_0)} \mapsto \gamma(1)$. If the action of T on M is not free, the same argument works by replacing the fundamental group of M/T at p_0 , by the corresponding orbifold fundamental group. \oslash

2.6 Alternative model

Next we present a model for (M, σ) , up to T -equivariant symplectomorphisms, which does not involve the universal cover of M/T , but rather a smaller cover of M/T .

Following Borzellino's article [7, Def. 6] we make the following definition.

Definition 2.44 Let \mathcal{O} be a smooth orbifold. The *first integral orbifold homology* $H_1^{\text{orb}}(\mathcal{O}, \mathbb{Z})$ of a smooth orbifold \mathcal{O} is defined as the abelianization of the orbifold fundamental group $\pi_1^{\text{orb}}(\mathcal{O}, x)$ of \mathcal{O} at x , where x is any point in \mathcal{O} . \oslash

Remark 2.45 The definition of $H_1^{\text{orb}}(\mathcal{O}, \mathbb{Z})$ does not depend on the choice of the point $x \in \mathcal{O}$ in the sense that all abelianizations of $\pi_1^{\text{orb}}(\mathcal{O}, x)$, $x \in \mathcal{O}$, are naturally identified with each other. \oslash

The Hurewicz map at the level of smooth orbifolds may be defined analogously to the usual Hurewicz map at the level of smooth manifolds. Indeed, if \mathcal{O} is a smooth orbifold, the *orbifold Hurewicz map* h_1 , from the orbifold fundamental group $\pi_1^{\text{orb}}(\mathcal{O}, x_0)$ to the first orbifold homology group $H_1^{\text{orb}}(\mathcal{O}, \mathbb{Z})$, assigns to a homotopy class of a loop based at x_0 , c.f. Hatcher's [21, Sec. 4.2] or Spanier's [51, Sec. 7.4], which is also a one-dimensional cycle, the homology class of that cycle. The mapping h_1 is a homomorphism from $\pi_1^{\text{orb}}(\mathcal{O}, x_0)$ to $H_1^{\text{orb}}(\mathcal{O}, \mathbb{Z})$.

Definition 2.46 Let (M, σ) be a compact, connected, symplectic manifold equipped with an effective symplectic action of a torus T for which at least one, and hence every T -orbit is a $\dim T$ -dimensional symplectic submanifold of (M, σ) , and let h_1 from $\pi_1^{\text{orb}}(M/T, p_0)$ to $H_1^{\text{orb}}(M/T, \mathbb{Z})$ be the orbifold Hurewicz mapping. There exists a unique homomorphism $H_1^{\text{orb}}(M/T, \mathbb{Z}) \rightarrow T$, which we call μ_h , such that

$$\mu = \mu_h \circ h_1, \tag{2.25}$$

where $\mu: \pi_1^{\text{orb}}(M/T, p_0) \rightarrow T$ is the monodromy homomorphism of the connection $\Omega = \{\Omega_x = (T_x(T \cdot x))^{\sigma_x}\}$ of symplectic orthogonal complements to the tangent spaces to the T -orbits in M (c.f. Proposition 2.17). The homomorphism μ_h is independent of the choice of base point p_0 . \oslash

See Remark 2.37 for a description of the ingredients involved in the following theorem.

Theorem 2.47. *Let (M, σ) be a compact, connected, symplectic manifold equipped with an effective symplectic action of a torus T for which at least one, and hence every T -orbit, is a $\dim T$ -dimensional symplectic submanifold of (M, σ) . Then there exist a $(\dim M - \dim T)$ -dimensional symplectic manifold (N, σ^N) , a commutative group Δ which acts properly on N with finite stabilizers and such that N/Δ is compact, and a group monomorphism $\mu'_h: \Delta \rightarrow T$, such that the symplectic T -manifold (M, σ) is T -equivariantly symplectomorphic to $N \times_{\Delta} T$, where Δ acts on $N \times T$ by the diagonal action $x(y, t) = (x * y^{-1}, \mu'_h(x) \cdot t)$, where $*: \Delta \times N \rightarrow N$ denotes the action of Δ on N . $N \times_{\Delta} T$ is equipped with the action of T by translations which descends from the action of T by translations on the right factor of the product $N \times T$, and the symplectic form induced on the quotient by the product symplectic form $\sigma^N \oplus \sigma^T$ on the Cartesian product $N \times T$. Here σ^T is the unique translation invariant symplectic form on T induced by the anti-symmetric bilinear form σ^t , where $\sigma_x(X_M(x), Y_M(x)) = \sigma^t(X, Y)$ for every $X, Y \in \mathfrak{t}$, and every $x \in M$, and σ^N is inherited from the symplectic form ν on the orbit space M/T , c.f. Lemma 2.28 by means of the covering map $N \rightarrow M/T$, c.f. Remark 2.48. The structure of smooth manifold for N is inherited from the smooth manifold structure of the orbifold universal cover $\widetilde{M/T}$, since N is defined as the quotient $\widetilde{M/T}/K$ where K is the kernel of the monodromy homomorphism $\mu: \pi_1^{\text{orb}}(M/T, p_0) \rightarrow T$. Moreover, N is a regular covering of the orbifold M/T with covering group Δ .*

Proof. We proved in Theorem 2.36 that for any element $[\gamma] \in \widetilde{M/T}$ and $t \in T$, the mapping defined by $\Phi([\gamma], t) := t \cdot \lambda_{\widetilde{\gamma}}(1) \in M$, induces a T -equivariant symplectomorphism ϕ from the associated bundle $\widetilde{M/T} \times_{\pi_1^{\text{orb}}(M/T, p_0)} T$ onto the symplectic T -manifold (M, σ) . See Definition 2.32 for the construction of the symplectic form and T -action on this associated bundle. Let K be the kernel subgroup of the monodromy homomorphism μ from the orbifold fundamental group $\pi_1^{\text{orb}}(M/T, p_0)$ into the torus T . The kernel subgroup K is a normal subgroup of $\pi_1^{\text{orb}}(M/T, p_0)$ which contains the commutator subgroup C of $\pi_1^{\text{orb}}(M/T, p_0)$. There is a unique homomorphism $\mu_c: \pi_1^{\text{orb}}(M/T, p_0)/C \rightarrow T$ such that $\mu = \mu_c \circ \chi$, where χ is the quotient homomorphism from the orbifold fundamental group $\pi_1^{\text{orb}}(M/T, p_0)$ onto $\pi_1^{\text{orb}}(M/T, p_0)/C$. The orbifold Hurewicz map h_1 , from the orbifold fundamental group $\pi_1^{\text{orb}}(M/T, p_0)$ to the first orbifold homology group $H_1^{\text{orb}}(M/T, \mathbb{Z})$, assigns to a homotopy class of a loop based at p_0 , which is also a one-dimensional cycle, the homology class of that cycle, is a homomorphism from $\pi_1^{\text{orb}}(M/T, p_0)$ onto $H_1^{\text{orb}}(M/T, \mathbb{Z})$ with kernel the commutator subgroup C . Therefore the quotient group $K/C \leq \pi_1^{\text{orb}}(M/T, p_0)/C$ can be viewed as a subgroup of the first orbifold homology group

$$H_1^{\text{orb}}(M/T, \mathbb{Z}) \simeq \pi_1^{\text{orb}}(M/T, p_0)/C,$$

where the symbol \simeq stands for the projection induced by the Hurewicz map h_1 from the orbifold fundamental group $\pi_1(M/T, p_0)$ into the first orbifold homology group $H_1^{\text{orb}}(M/T, \mathbb{Z})$. Let $N := \widetilde{M/T}/K$. By Lemma 2.22 and Remark 2.37, N is a smooth manifold (diffeomorphic to the integral manifold of the distribution Ω , c.f. Proposition 2.17). Because the universal cover of M/T is a regular orbifold covering of which the orbifold fundamental group is the covering group, the quotient N is a regular orbifold covering of the orbit space M/T with covering group

$$\Delta := H_1^{\text{orb}}(M/T, \mathbb{Z})/(K/C) \simeq \pi_1^{\text{orb}}(M/T, p_0)/C.$$

The group Δ is a commutative group. Let Δ acts on the torus T by the mapping

$$(x, t) \mapsto \mu'_h(x) t, \quad (2.26)$$

where the mapping $\mu'_h: \Delta \rightarrow T$ is the quotient homomorphism induced by μ_h and π , where recall that μ_h denotes the homomorphism induced on homology from the monodromy μ associated to the connection Ω , c.f. Proposition 2.17. The mapping μ'_h is injective because the subgroup K equals the kernel of μ . Our construction produces an identification of the bundle $\widetilde{M/T} \times_{\pi_1^{\text{orb}}(M/T, p_0)} T$ and $N \times_{\Delta} T$, where Δ is acting by the diagonal action on $N \times T$ to give rise to $N \times_{\Delta} T$ (c.f expression (2.26)), which intertwines the actions of T by translation on the right T -factor of both spaces. In this way the mapping ϕ induces a diffeomorphism from the associated bundle $N \times_{\Delta} T$ to M , which intertwines the action of T by translations on the right factor of $N \times_{\Delta} T$ with the action of T on M . By the same proof as in Theorem 2.41, the T -equivariant diffeomorphism ϕ from $N \times_{\Delta} T$ onto M pulls back the symplectic form on M to the symplectic form on $N \times_{\Delta} T$ given in Remark 2.48. \square

Remark 2.48 This remark justifies why the symplectic form on the model space defined in Theorem 2.47 is correctly defined. Let us assume the terminology used in the statement of Theorem 2.47. The pull-back of the 2-form ν on M/T to the smooth manifold N by means of the covering map ϕ (c.f. Lemma 2.28) such that $\pi^*\nu|_{\Omega_x} = \sigma|_{\Omega_x}$ for every $x \in M$, where $\Omega := \{\Omega_x\}_{x \in M}$ is the distribution on M of symplectic orthogonal complements to the tangent spaces to the T -orbits, defined by Proposition 2.17, is a Δ -invariant symplectic form on N . The symplectic form on T determined by the antisymmetric bilinear form σ^t given by Lemma 2.1 is translation invariant, and therefore Δ -invariant. The direct sum of the symplectic form on N and the symplectic form on T is a Δ -invariant and T -invariant symplectic form on $N \times T$, and therefore there is a unique symplectic form on $N \times_{\Delta} T$ of which the pull-back by the covering map $N \times T \rightarrow N \times_{\Delta} T$ is equal to the given symplectic form on $N \times T$. \oslash

3 Classification: free case

Throughout this section (M, σ) is a compact and connected symplectic manifold and T is a torus which acts freely on (M, σ) by means of symplectomorphisms, and such that at least one T -orbit is a $\dim T$ -dimensional symplectic submanifold of (M, σ) . Our goal is to use the model for (M, σ) which we constructed in Definition 2.32 to provide a classification of (M, σ) when $\dim T = \dim M - 2$, in terms of a collection of invariants.

3.1 Monodromy Invariant

We define what we call the free monodromy invariant of (M, σ) , ingredient 4) in Definition 3.9.

3.1.1 Intersection Forms and Geometric Maps

Let Σ be a compact, connected, orientable, smooth surface of genus g , where g is a positive integer. Recall the *algebraic intersection number*

$$\cap: H_1(\Sigma, \mathbb{Z}) \otimes H_1(\Sigma, \mathbb{Z}) \rightarrow \mathbb{Z}, \quad (3.1)$$

which extends uniquely to the *intersection form*

$$\cap: H_1(\Sigma, \mathbb{R}) \otimes H_1(\Sigma, \mathbb{R}) \rightarrow \mathbb{R}, \quad (3.2)$$

which turns $H_1(\Sigma, \mathbb{R})$ into a symplectic vector space. It is always possible to find, c.f. [21, Ex. 2A.2], a so called “symplectic” basis, in the sense of [37, Th. 2.3].

Definition 3.1 Let Σ be a compact, connected, orientable, smooth surface of genus g , where g is a positive integer. A collection of elements $\alpha_i, \beta_i, 1 \leq i \leq g$, of $H_1(\Sigma, \mathbb{Z}) \subset H_1(\Sigma, \mathbb{R})$ such that

$$\alpha_i \cap \alpha_j = \beta_i \cap \beta_j = 0, \quad \alpha_i \cap \beta_j = \delta_{ij} \quad (3.3)$$

for all i, j with $1 \leq i, j \leq g$ is called a *symplectic basis of the group $H_1(\Sigma, \mathbb{Z})$* or a *symplectic basis of the symplectic vector space $(H_1(\Sigma, \mathbb{R}), \cap)$* . \oslash

Hence the matrix associated to the antisymmetric bilinear form \cap on the basis α_i, β_i is the block diagonal matrix

$$J_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ & & 0 & 1 \\ & & -1 & 0 \\ & & & \ddots \\ & & & & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix}.$$

Notice that the symplectic linear group $Sp(2g, \mathbb{R}) \subset GL(2g, \mathbb{R})$ is the group of matrices A such that

$$A \cdot J_0 \cdot A^t = J_0, \quad (3.4)$$

and therefore it is natural to denote by $Sp(2g, \mathbb{Z})$ the group of matrices $A \in GL(2g, \mathbb{Z})$ which satisfy expression (3.4).

Remark 3.2 The group $Sp(2g, \mathbb{Z})$ is also called Siegel’s modular group, and denoted by Γ_g , see for example J. Birman’s article [5]. Generators for $Sp(2g, \mathbb{Z})$ were first determined by L. K. Hua and I. Reiner [23]. Later H. Klingen found a characterization of [29] for $g \geq 2$ by a finite system of relations. Birman’s article [5] reduces Klingen’s article to a more usable form, in which she explicitly describes the calculations in Klingen’s paper, among other results. I thank J. McCarthy for making me aware of J. Birman’s article and for pointing me to his article [35] which contains a generalization of it. \oslash

Remark 3.3 Let g be a non-negative integer, and let Σ be a compact, connected, oriented, smooth surface of genus g . If the first homology group $H_1(\Sigma, \mathbb{Z})$ is identified with $GL(2g, \mathbb{Z})$ by means of a choice of symplectic basis, the group of automorphisms of $H_1(\Sigma, \mathbb{Z})$ which preserve the intersection form gets identified with the group $Sp(2g, \mathbb{Z})$ of matrices in $GL(2g, \mathbb{Z})$ which satisfy expression (3.4). Indeed, let α_i, β_i be a symplectic basis of $H_1(\Sigma, \mathbb{Z})$ (i.e. whose elements satisfy expression (3.3)). The $(2g \times 2g)$ -matrix $M(\alpha_i, \beta_i, f)$ of f with respect to the basis α_i, β_i is an element of the linear group $GL(2g, \mathbb{Z})$ of $(2g \times 2g)$ -invertible matrices with integer coefficients and it is an exercise to check that f preserves the intersection form on $H_1(\Sigma, \mathbb{R})$ if and only if $M(\alpha_i, \beta_i, f) \cdot J_0 \cdot M(\alpha_i, \beta_i, f)^t = J_0$. \oslash

An orientation preserving diffeomorphism induces an isomorphism in homology which preserves the intersection form. Moreover, the converse also holds, c.f. [33, pp. 355–356]. An algebraic proof of this result is given by J. Birman in [5, pp. 66–67] as a consequence of [5, Thm. 1] (see also the references therein). To be self contained we present a sketch of proof here, following a preliminary draft by B. Farb and D. Margalit [14]. Denote by $MCG(\Sigma)$ the mapping class group of orientation preserving diffeomorphisms of Σ modulo isotopies. After a choice of basis, there is a natural homomorphism

$$\mu: MCG(\Sigma) \rightarrow Sp(2g, \mathbb{Z}). \quad (3.5)$$

Lemma 3.4. *Let g be a non-negative integer, and let Σ, Σ' be compact, connected, oriented, smooth surfaces of the same genus g . Then a group isomorphism f from the first homology group $H_1(\Sigma, \mathbb{Z})$ onto the first homology group $H_1(\Sigma', \mathbb{Z})$ preserves the intersection form if and only if there exists an orientation preserving diffeomorphism $i: \Sigma \rightarrow \Sigma'$ such that $f = i_*$. Moreover, if $\{U_k\}_{k=1}^m \subset \Sigma, \{U'_k\}_{k=1}^m \subset \Sigma'$ are finite disjoint collections of embedded disks, i can be chosen to map U_k to U'_k for all $k, 1 \leq k \leq m$.*

Proof. W.l.o.g. assume that $\Sigma = \Sigma'$. It is immediate that an orientation preserving diffeomorphism of Σ induces an intersection form preserving automorphism of $H_1(\Sigma, \mathbb{Z})$. Let γ be the permutation of $\{1, \dots, 2g\}$ which transposes $2i$ and $2i - 1$ for $1 \leq i \leq 2g$. We define the ij^{th} elementary symplectic matrix by

$$\Sigma^{ij} = \begin{cases} I_{2g} + E_{ij} & \text{if } i = \gamma(j); \\ I_{2g} + E_{ij} - (-1)^{i+j} E_{\gamma(i)\gamma(j)} & \text{otherwise.} \end{cases}$$

where I_{2g} stands for the identity matrix of dimension $2g$ and E_{ij} is the matrix with a 1 in the ij^{th} position and 0's elsewhere. It is a classical fact that $Sp(2g, \mathbb{Z})$ is generated by the matrices Σ^{ij} . To prove the lemma is equivalent to showing that μ in (3.5) is surjective onto $Sp(2g, \mathbb{Z})$. Let τ_b be the Dehn twist about a simple closed curve b . Then for integer values of k , the image $\mu(\tau_b^k)$ is given by

$$a \mapsto a + k \cdot \cap(a, b) b. \quad (3.6)$$

We may restrict our attention to a subsurface Σ_0 of Σ of genus 1 in the case of $i = \gamma(j)$, or of genus 2 otherwise, such that the i^{th} and j^{th} basis elements are supported on it, as well as assume

that i is odd, and that $H_1(\Sigma_0, \mathbb{Z})$ is spanned by a symplectic basis $\alpha_1, \beta_1, \alpha_2, \beta_2$. Then using that μ is a homomorphism and (3.6) one shows $\mu(\tau_{\alpha_1}^{-1}) = \Sigma^{1,2}$, $\mu(\tau_{\alpha_2}^{-1}\tau_{\alpha_1}^{-1}\tau_{\alpha_1+\beta_2}) = \Sigma^{1,3}$ and $\mu(\tau_{\alpha_2}\tau_{\alpha_1}\tau_{\alpha_1+\beta_2}^{-1}) = \Sigma^{3,2}$. \square

3.1.2 Construction

In this section we construct the monodromy invariant of our symplectic manifold (M, σ) , c.f. Definition 3.8. The fact that such invariant is well defined uses only linear algebra. To any group homomorphism

$$f: H_1(\Sigma, \mathbb{Z}) \rightarrow T,$$

we can assign the $2g$ -tuple $(f(\alpha_i), f(\beta_i))_{i=1}^g$, where $\alpha_i, \beta_i, 1 \leq i \leq g$ is a basis of the homology group $H_1(\Sigma, \mathbb{Z})$ satisfying formulas (3.3). Conversely, given a $2g$ -tuple $(a_1, b_1, \dots, a_g, b_g) \in T^{2g}$, the commutativity of T implies, by the universal property of free abelian groups [13, Ch. I.3] that there exists a unique group homomorphism from the homology group $H_1(\Sigma, \mathbb{Z})$ into the torus T which sends α_i to a_i and β_j to b_j , for all values of i, j with $1 \leq i, j \leq g$. Notice that this tuple depends on the choice of basis. For each such basis we have an isomorphism of groups

$$f \mapsto (f(\alpha_i), f(\beta_i))_{i=1}^g, \quad (3.7)$$

between the homomorphism group $\text{Hom}(H_1(\Sigma, \mathbb{Z}), T)$ and the Cartesian product T^{2g} .

Definition 3.5 Let T be a torus and m an even positive integer. Let \mathcal{H} be a subgroup of $\text{GL}(m, \mathbb{Z})$. A matrix A in \mathcal{H} acts on the Cartesian product T^m by sending an m -tuple x to $x \cdot A^{-1}$ by identifying x with a homomorphism from \mathbb{Z}^{2g} to T . We say that two m -tuples $x, y \in T^m$ are \mathcal{H} -equivalent if they lie in the same \mathcal{H} -orbit, i.e. if there exists a matrix $A \in \mathcal{H}$ such that $y = x \cdot A$. We write $\mathcal{H} \cdot x$ for the \mathcal{H} -orbit of x , and T^m/\mathcal{H} for the set of all \mathcal{H} -orbits. \oslash

Lemma 3.6. *Let T be an abelian group, let Σ be a compact, connected, smooth orientable surface, let α_i, β_i and α'_i, β'_i be symplectic bases of the integral homology group $H_1(\Sigma, \mathbb{Z})$, and let $f: H_1(\Sigma, \mathbb{Z}) \rightarrow T$ be a group homomorphism. Then there exists a matrix in the symplectic group $\text{Sp}(2g, \mathbb{Z})$ which takes the image tuple $(f(\alpha_i), f(\beta_i))$ to the image tuple $(f(\alpha'_i), f(\beta'_i))$.*

Proof. Because any two symplectic bases α_i, β_i and α'_i, β'_i of the group $H_1(\Sigma, \mathbb{Z})$ are taken onto each other by an element of the symplectic group $\text{Sp}(2g, \mathbb{Z})$, the change of basis matrix from the basis α_i, β_i , to the basis α'_i, β'_i is in the group $\text{Sp}(2g, \mathbb{Z})$. Notice that the $(2j-1)^{\text{th}}$ -column of this matrix consists of the coordinates of α'_j , with respect to the basis α'_i, β'_i , and its $(2j)^{\text{th}}$ -column consists of the coordinates of β'_j , and hence we have that the tuple $(f(\alpha_i), f(\beta_i))_{i=1}^g$ is obtained by applying such matrix to the tuple $(f(\alpha'_i), f(\beta'_i))_{i=1}^g$. \square

Lemma 3.6 shows that the assignment

$$f \mapsto \text{Sp}(2g, \mathbb{Z}) \cdot (f(\alpha_i), f(\beta_i))_{i=1}^g \quad (3.8)$$

induced by expression (3.7) is well defined independently of the choice of basis $\alpha_i, \beta_i, 1 \leq i \leq g$ of $H_1(\Sigma, \mathbb{Z})$, as long as it is a symplectic basis. The following is a consequence of Lemma 3.6 and Definition 3.5.

Lemma 3.7. Let (M, σ) be a compact, connected $2n$ -dimensional symplectic manifold equipped with a free symplectic action of a $(2n - 2)$ -dimensional torus T , for which at least one, and hence every T -orbit is a $(2n - 2)$ -dimensional symplectic submanifold of (M, σ) . If α_i, β_i and α'_i, β'_i , $1 \leq i \leq g$, are symplectic bases of $H_1(M/T, \mathbb{Z})$, then for every homomorphism $f: H_1(M/T, \mathbb{Z}) \rightarrow T$, we have that the $\mathrm{Sp}(2g, \mathbb{Z})$ -orbits of the tuples $(f(\alpha_i), f(\beta_i))_{i=1}^g$ and $(f(\alpha'_i), f(\beta'_i))_{i=1}^g$ are equal.

Definition 3.8 Let (M, σ) be a compact, connected, $2n$ -dimensional symplectic manifold equipped with a free symplectic action of a torus T for which at least one, and hence every T -orbit is a $(2n - 2)$ -dimensional symplectic submanifold of (M, σ) . The *free monodromy invariant* of (M, σ, T) is the $\mathrm{Sp}(2g, \mathbb{Z})$ -orbit

$$\mathrm{Sp}(2g, \mathbb{Z}) \cdot (\mu_h(\alpha_i), \mu_h(\beta_i))_{i=1}^g \in T^{2g}/\mathrm{Sp}(2g, \mathbb{Z}),$$

where α_i, β_i , $1 \leq i \leq g$, is a basis of the homology group $H_1(M/T, \mathbb{Z})$ satisfying expression (3.3) and μ_h is the homomorphism induced on homology by the monodromy homomorphism μ of the connection of symplectic orthogonal complements to the tangent spaces to the T -orbits (c.f. Proposition 2.17) by means of the Hurewicz map (c.f. Definition 3.5, formula (2.25)). \circledcirc

3.2 Uniqueness

3.2.1 List of ingredients of (M, σ, T)

We start with the following definition.

Definition 3.9 Let (M, σ) be a compact connected $2n$ -dimensional symplectic manifold equipped with a free symplectic action of a $(2n - 2)$ -dimensional torus T for which at least one, and hence every T -orbit is a $(2n - 2)$ -dimensional symplectic submanifold of (M, σ) . The *list of ingredients* of (M, σ, T) consists of the following items.

- 1) The genus g of the surface M/T (c.f. Remark 2.14).
- 2) The total symplectic area of the symplectic surface $(M/T, \nu)$, where the symplectic form ν is defined by the condition $\pi^*\nu|_{\Omega_x} = \sigma|_{\Omega_x}$ for every $x \in M$, $\pi: M \rightarrow M/T$ is the projection map, and where for each $x \in M$, $\Omega_x = (T_x(T \cdot x))^{\sigma_x}$ (c.f. Lemma 2.28).
- 3) The non-degenerate antisymmetric bilinear form $\sigma^t: \mathfrak{t} \times \mathfrak{t} \rightarrow \mathbb{R}$ on the Lie algebra \mathfrak{t} of T such that for all $X, Y \in \mathfrak{t}$ and all $x \in M$ $\sigma_x(X_M(x), Y_M(x)) = \sigma^t(X, Y)$ (c.f. Lemma 2.1).
- 4) The free monodromy invariant of (M, σ, T) , i.e. the $\mathrm{Sp}(2g, \mathbb{Z})$ -orbit

$$\mathrm{Sp}(2g, \mathbb{Z}) \cdot (\mu_h(\alpha_i), \mu_h(\beta_i))_{i=1}^g$$

of the $2g$ -tuple $(\mu_h(\alpha_i), \mu_h(\beta_i))_{i=1}^g \in T^{2g}$, c.f. Definition 3.8. \circledcirc

3.2.2 Uniqueness Statement

The next two results say that the list of ingredients of (M, σ, T) as in Definition 3.9 is a complete set of invariants of (M, σ, T) . We start with a preliminary remark.

Remark 3.10 Let T be a torus. Let $x_0 \in X$. Let $p: X \rightarrow B$, $p': X' \rightarrow B'$ be smooth principal T -bundles, equipped with flat connections Ω , Ω' , and let $\Phi: X \rightarrow X'$ be a T -bundle isomorphism such that $\Phi^*\Omega' = \Omega$ (i.e. $\Omega'_{\Phi(x)} = T_x \Phi(\Omega_x)$ for each $x \in X$). Let $x'_0 := \Phi(x_0)$, and let $\widehat{\Phi}: B \rightarrow B'$ be induced by Φ and such that $\widehat{\Phi}(b_0) = b'_0$, where $p(x_0) = b_0$, $p'(x'_0) = b'_0$. The monodromy homomorphism $\mu: \pi_1(B, b_0) \rightarrow T$ associated to the connection Ω is the unique homomorphism such that

$$\lambda_\gamma(1) = \mu([\gamma]) \cdot x_0, \quad (3.9)$$

for every path $\gamma: [0, 1] \rightarrow B$ such that $\gamma(0) = \gamma(1) = b_0$, where λ_γ is the unique horizontal lift of γ with respect to the connection Ω , such that $\lambda_\gamma(0) = x_0$.

Applying Φ to both sides of expression (3.9), and using that Φ preserves the horizontal subspaces, denoting by $\lambda'_{\gamma'}$ the unique horizontal lift with respect to Ω' , of any loop $\gamma': [0, 1] \rightarrow B'$ with $\gamma'(0) = \gamma'(1) = b'_0$, we obtain that $\lambda'_{\widehat{\Phi}\gamma}(1) = (\mu \circ (\widehat{\Phi}_*)^{-1})([\widehat{\Phi}\gamma]) \cdot x'_0$, where $\widehat{\Phi}_*: \pi_1(B, b_0) \rightarrow \pi_1(B', b'_0)$ is the isomorphism induced by $\widehat{\Phi}$, which by the uniqueness of the monodromy homomorphism, implies that $\mu' = \mu \circ (\widehat{\Phi}_*)^{-1}: \pi_1(B', b'_0) \rightarrow T$, and since $\mu = \mu_h \circ h_1$ and $\mu' = \mu_h \circ h_1$, see expression (2.25), that $\mu'_h = \mu_h \circ (\widehat{\Phi}_*)^{-1}: \pi_1(B', b'_0) \rightarrow T$, where in this case $\widehat{\Phi}_*: H_1(B, \mathbb{Z}) \rightarrow H_1(B', \mathbb{Z})$ is the homomorphism induced by $\widehat{\Phi}$ in homology. \circlearrowright

Lemma 3.11. *Let (M, σ) be a compact connected $2n$ -dimensional symplectic manifold equipped with a free symplectic action of a $(2n - 2)$ -dimensional torus T for which at least one, and hence every T -orbit is a $(2n - 2)$ -dimensional symplectic submanifold of (M, σ) . If (M', σ') is a compact connected $2n$ -dimensional symplectic manifold equipped with a free symplectic action of T for which at least one, and hence every T -orbit is a $(2n - 2)$ -dimensional symplectic submanifold of (M', σ') , and (M', σ') is T -equivariantly symplectomorphic to (M, σ) , then the list of ingredients of (M', σ', T) is equal to the list of ingredients of (M, σ, T) .*

Proof. Let Φ be a T -equivariant symplectomorphism from (M, σ) onto (M', σ') . Like in the proof of Lemma 2.29, the mapping Φ descends to a symplectomorphism $\widehat{\Phi}$ from the orbit space $(M/T, \nu)$ onto the orbit space $(M'/T, \nu')$. By Remark 2.14, the orbit spaces M/T and M'/T are compact, connected, orientable, smooth surfaces, and because they are diffeomorphic, M/T and M'/T must have the same genus g .

Lemma 2.29 and Remark 2.30 imply that ingredient 2) of (M, σ) equals ingredient 2) of (M', σ') .

If $X, Y \in \mathfrak{t}$, then the T -equivariance of Φ implies that $\Phi^*(X_{M'}) = X_M$, $\Phi^*(Y_{M'}) = Y_M$. In combination with $\sigma = \widehat{\Phi}^*\sigma'$, this implies, in view of Lemma 2.1, that

$$\sigma^t(X, Y) = \Phi^*(\sigma'(X_{M'}, Y_{M'})) = \Phi^*((\sigma')^t(X, Y)) = (\sigma')^t(X, Y),$$

where we have used in the last equation that $(\sigma')^t(X, Y)$ is a constant on M' . This proves that $\sigma^t = (\sigma')^t$. Since $\Phi^*\Omega' = \Omega$, we have that

$$\mu'_h = \mu_h \circ (\widehat{\Phi}_*)^{-1},$$

as mappings from the orbifold homology group $H_1(M'/T, \mathbb{Z})$ into the torus T , where the mapping $\widehat{\Phi}_*$ from the homology group $H_1(M/T, \mathbb{Z})$ to $H_1(M'/T, \mathbb{Z})$ is the group isomorphism induced by the orbifold diffeomorphism $\widehat{\Phi}$ from the orbit space M/T onto M'/T (see Remark 3.10). By Lemma 3.4, $\widehat{\Phi}_*$ preserves the intersection-form, and therefore the images under $\widehat{\Phi}_*$ of $\alpha_i, \beta_i, 1 \leq i \leq g$, which we will call α'_i, β'_i , form a symplectic basis of $H_1(M'/T, \mathbb{Z})$. Hence $\mu_h(\alpha_i) = \mu'_h(\alpha'_i)$ and $\mu_h(\beta_i) = \mu'_h(\beta'_i)$, which in turn implies that ingredient 4) of (M, σ) equals ingredient 4) of (M', σ') . \square

Proposition 3.12. *Let (M, σ) be a compact connected $2n$ -dimensional symplectic manifold equipped with a free symplectic action of a $(2n - 2)$ -dimensional torus T for which at least one, and hence every T -orbit is a $(2n - 2)$ -dimensional symplectic submanifold of (M, σ) . Then if (M', σ') is a compact connected $2n$ -dimensional symplectic manifold equipped with a free symplectic action of T for which at least one, and hence every T -orbit is a $(2n - 2)$ -dimensional symplectic submanifold of (M', σ') , and the list of ingredients of (M, σ, T) is equal to the list of ingredients of (M', σ', T) , then (M, σ) is T -equivariantly symplectomorphic to (M', σ') .*

Proof. Suppose that the list of ingredients of (M, σ) equals the list of ingredients of (M', σ') . Let α_i, β_i , and α'_i, β'_i be, respectively, symplectic bases of the first integral homology groups $H_1(M/T, \mathbb{Z})$ and $H_1(M'/T, \mathbb{Z})$, and suppose that ingredient 4) of (M, σ) is equal to ingredient 4) of (M', σ') . Let μ, μ', μ_h, μ'_h be the corresponding homomorphisms respectively associated to $(M, \sigma), (M', \sigma')$ as in Definition 2.46. Then, by Definition 3.5, there exists a matrix in the integer symplectic group $Sp(2g, \mathbb{Z})$ which takes the tuple of images of the α_i, β_i under μ_h to the tuple of images of α'_i, β'_i under μ'_h , which means that

$$\mu_h = \mu'_h \circ G, \tag{3.10}$$

as maps from the homology group $H_1(M/T, \mathbb{Z})$ to the torus T , where the mapping G is the intersection form preserving automorphism from the homology group $H_1(M/T, \mathbb{Z})$ to $H_1(M'/T, \mathbb{Z})$ whose matrix with respect to the bases α_i, β_i and α'_i, β'_i is precisely the aforementioned matrix. Since M/T and M'/T have the same genus, by Lemma 3.4, there exists a surface diffeomorphism $F: M/T \rightarrow M'/T$ such that $F_* = G$, and hence by (3.10),

$$\mu = \mu' \circ F_*. \tag{3.11}$$

Let ν and ν' be the symplectic forms given by Lemma 2.28. If $\nu_0 := F^*\nu'$, the symplectic manifold $(M/T, \nu_0)$ is symplectomorphic to $(M'/T, \nu')$, by means of F . Let $\widetilde{\nu_0}$ be the pullback of the 2-form ν_0 by the universal cover $\psi: \widetilde{M}/T \rightarrow M/T$ of M/T based at p_0 , and similarly we define $\widetilde{\nu}'$ by means of the universal cover $\psi': \widetilde{M}'/T \rightarrow M'/T$ based at $F(p_0)$. Choose x_0, x'_0 such that $p_0 = \psi(x_0)$, $p'_0 = \psi'(x'_0)$. By standard covering space theory, the symplectomorphism F

between $(M/T, \nu_0)$ and $(M'/T, \nu')$ lifts to a unique symplectomorphism \tilde{F} between $(\widetilde{M/T}, \widetilde{\nu_0})$ and $(\widetilde{M'/T}, \widetilde{\nu'})$ such that $\tilde{F}(x_0) = x'_0$. Now, let σ^T be the unique translation invariant symplectic form on the torus T which is uniquely determined by the antisymmetric bilinear form σ^t , which since ingredient 3) of (M, σ) is equal to ingredient 3) of (M', σ') , equals the antisymmetric bilinear form $(\sigma')^t$. Then the product $\widetilde{M/T} \times T$ is T -equivariantly symplectomorphic to the product $\widetilde{M'/T} \times T$ by means of the map

$$([\gamma], t) \mapsto (\tilde{F}([\gamma]), t), \quad (3.12)$$

where T is acting by translations on the right factor of both spaces, $\widetilde{M/T} \times T$ is equipped with the symplectic form $\widetilde{\nu_0} \oplus \sigma^T$, and $\widetilde{M'/T} \times T$ is equipped with the symplectic form $\widetilde{\nu'} \oplus \sigma^T$. As in Definition 2.32, let $[\delta] \in \pi_1(M/T, p_0)$ act diagonally on the product $\widetilde{M/T} \times T$ by sending the pair $([\gamma], t)$ to $([\gamma \delta^{-1}], \mu([\delta]) t)$, and similarly let $[\delta'] \in \pi_1(M'/T, p_0)$ act on $\widetilde{M'/T} \times T$ by sending $([\gamma'], t)$ to $([\gamma' (\delta')^{-1}], \mu'([\delta']) t)$, hence giving rise to the well-defined quotient spaces $\widetilde{M/T} \times_{\pi_1(M/T, p_0)} T$ and $\widetilde{M'/T} \times_{\pi_1(M'/T, p'_0)} T$. (Here μ and μ' are the monodromy homomorphisms respectively associated to the connections Ω and Ω'). Because the based fundamental groups $\pi_1(M/T, p_0)$ and $\pi_1(M'/T, p'_0)$ act properly and discontinuously, both the action of T on the products, as well as the symplectic form, induce well-defined actions and symplectic forms on these quotients. Therefore, because of expression (3.11), the assignment induced by the product mapping (3.12)

$$\Psi: [[\gamma], t]_{\pi_1(M/T, p_0)} \mapsto [\tilde{F}([\gamma]), t]_{\pi_1(M'/T, p'_0)},$$

is a T -equivariant symplectomorphism between the associated bundle $\widetilde{M/T} \times_{\pi_1(M/T, p_0)} T$ and the associated bundle $\widetilde{M'/T} \times_{\pi_1(M'/T, p'_0)} T$. Because M/T and M'/T have the same genus and symplectic area, by Moser's theorem, c.f. [37, Th. 3.17], the (compact, connected, smooth, orientable) surfaces $(M/T, \nu)$ and $(M'/T, \nu')$ are symplectomorphic, and hence $(M/T, \nu)$ is symplectomorphic to $(M/T, \nu_0)$. By Theorem 2.36, (M, σ) is T -equivariantly symplectomorphic to the associated bundle $\widetilde{M/T} \times_{\pi_1(M/T, p_0)} T$, with the symplectic form $\widetilde{\nu} \oplus \sigma^T$, by means of a T -equivariant symplectomorphism ϕ , and hence T -equivariantly symplectomorphic to the associated bundle $\widetilde{M/T} \times_{\pi_1(M/T, p_0)} T$ with the symplectic form $\widetilde{\nu_0} \oplus \sigma^T$, say by means of $\tilde{\phi}$. Similarly, (M', σ') is T -equivariantly symplectomorphic to $\widetilde{M'/T} \times_{\pi_1(M'/T, p'_0)} T$, by means of a T -equivariant symplectomorphism φ , and therefore the composite $\varphi \circ \Psi \circ \tilde{\phi}^{-1}: M \rightarrow M'$ is a T -equivariant symplectomorphism between (M, σ) and (M', σ') . \square

3.3 Existence

3.3.1 List of ingredients for T

We start by making an abstract list of ingredients which we associate to a torus T .

Definition 3.13 Let T be a torus. The *list of ingredients for T* consists of the following items.

- i) A non-negative integer g .
- ii) A positive real number λ .
- iii) An non-degenerate antisymmetric bilinear form σ^t on the Lie algebra \mathfrak{t} of T .
- iv) A $\mathrm{Sp}(2g, \mathbb{Z})$ -orbit $\gamma \in T^{2g}/\mathrm{Sp}(2g, \mathbb{Z})$, where $\mathrm{Sp}(2g, \mathbb{Z})$ denotes the group of $2g$ -dimensional square symplectic matrices with integer entries, c.f. Definition 3.5.

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3.3.2 Existence Statement

Any list of ingredients as in Definition 3.13 gives rise to one of our manifolds with symplectic T -action.

Proposition 3.14. *Let T be a $(2n - 2)$ -dimensional torus. Then given a list of ingredients for T , as in Definition 3.13, there exists a symplectic $2n$ -dimensional manifold (M, σ) with a free symplectic action of T for which at least one, and hence every T -orbit is $(2n - 2)$ -dimensional symplectic submanifold of (M, σ) , and such that the list of ingredients of (M, σ, T) in Definition 3.9 is equal to the list of ingredients for T in Definition 3.13.*

Proof. Let \mathcal{I} be a list of ingredients for the torus T , as in Definition 3.13. Let the pair (Σ, σ^Σ) be a compact, connected symplectic surface of genus g given by ingredient i) of \mathcal{I} in Definition 3.13, and with total symplectic area equal to the positive real number λ given by ingredient ii) of \mathcal{I} . Let the space $\tilde{\Sigma}$ be the universal cover of the surface Σ based at an arbitrary regular point $p_0 \in \Sigma$, which we fix for the rest of the proof. Let T be the $(2n - 2)$ -dimensional torus that we started with, equipped with the unique T -invariant symplectic form σ^T on T whose associated non-degenerate antisymmetric bilinear form is $\sigma^t: \mathfrak{t} \times \mathfrak{t} \rightarrow \mathbb{R}$, given by ingredient iii) of \mathcal{I} . Write ingredient iv) of Definition 3.13 as $\gamma = \mathrm{Sp}(2g, \mathbb{Z}) \cdot (a_i, b_i)_{i=1}^g \in T^{2g}/\mathrm{Sp}(2g, \mathbb{Z})$, and recall the existence [21, Ex. 2A.1] of a symplectic \mathbb{Z} -basis α_i, β_i , $1 \leq i \leq g$, of the integral homology group $H_1(\Sigma, \mathbb{Z})$, satisfying expression (3.3). Let μ_h be the unique homomorphism from $H_1(\Sigma, \mathbb{Z})$ into T such that

$$\mu_h(\alpha_i) := a_i, \quad \mu_h(\beta_i) := b_i,$$

for all $1 \leq i \leq 2g$. Define $\mu := \mu_h \circ h_1$, where h_1 denotes the Hurewicz homomorphism from $\pi_1(\Sigma, p_0)$ to $H_1(\Sigma, \mathbb{Z})$. μ is a homomorphism from $\pi_1(\Sigma, p_0)$ into T . Let the fundamental group $\pi_1(\Sigma, p_0)$ act on the Cartesian product $\tilde{\Sigma} \times T$ by the diagonal action $[\delta] \cdot ([\gamma], t) = ([\delta \gamma^{-1}], \mu([\delta]) t)$. We equip the universal cover $\tilde{\Sigma}$ with the symplectic form $\sigma^{\tilde{\Sigma}}$ obtained as the pullback of σ^Σ by the universal covering mapping $\tilde{\Sigma} \rightarrow \Sigma$, and we equip the product space $\tilde{\Sigma} \times T$ with the product symplectic form $\sigma^{\tilde{\Sigma}} \oplus \sigma^T$, and let the torus T act by translations on the right factor of $\tilde{\Sigma} \times T$. Because $\pi_1(\Sigma, p_0)$ is acting properly and discontinuously, the symplectic form on $\tilde{\Sigma} \times T$ passes to a unique symplectic $\sigma_{\text{model}}^\Sigma := \widetilde{\sigma^{\tilde{\Sigma}} \oplus \sigma^T}$ form on M_{model}^Σ (as in the proof of Theorem 2.36).

Similarly, the action of the torus T by translations on the right factor of $\tilde{\Sigma} \times T$ passes to an action of T on M_{model}^Σ , which is free. Define the associated bundle

$$M_{\text{model}, p_0}^\Sigma := \tilde{\Sigma} \times_{\pi_1(\Sigma, p_0)} T. \quad (3.13)$$

It follows from the construction that $(M_{\text{model}}^\Sigma, \sigma_{\text{model}}^\Sigma)$ is a compact, connected symplectic manifold, with a free T -action for which every T -orbit is a $\dim T$ -dimensional symplectic submanifold of $(M_{\text{model}}^\Sigma, \sigma_{\text{model}}^\Sigma)$.

It is left to show that the list \mathcal{I} equals the list of ingredients of $(M_{\text{model}}^\Sigma, \sigma_{\text{model}}^\Sigma)$. Since the action of the torus T on M_{model}^Σ is induced by the action of T on the right factor of $\tilde{\Sigma} \times T$, $M_{\text{model}}^\Sigma/T$ is symplectomorphic to $\tilde{\Sigma}/\pi_1(\Sigma, p_0)$ with the symplectic form induced by the mapping $\tilde{\Sigma} \rightarrow \tilde{\Sigma}/\pi_1(\Sigma, p_0)$, which by construction (i.e. by regularity of the orbifold universal cover) is symplectomorphic to the quotient (Σ, σ^Σ) . Therefore $M_{\text{model}}^\Sigma/T$ with the symplectic form ν_Σ given by Lemma 2.28, is symplectomorphic to (Σ, σ^Σ) , and in particular the total symplectic area of $(M_{\text{model}}^\Sigma/T, \nu_{M_{\text{model}}^\Sigma}/T)$ equals the total symplectic area of (Σ, σ^Σ) , which is equal to the positive real number λ . Let $p: \tilde{\Sigma} \times T \rightarrow M_{\text{model}}$ be the projection map. It follows from the definition of $\sigma_{\text{model}}^\Sigma$ that for all $X, Y \in \mathfrak{t}$, the real number

$$(\sigma_{\text{model}}^\Sigma)_{[[\gamma], t]_{\pi_1(\Sigma, p_0)}}(X_{M_{\text{model}}^\Sigma}([[\gamma], t]_{\pi_1(\Sigma, p_0)}), Y_{M_{\text{model}}^\Sigma}([[\gamma], t]_{\pi_1(\Sigma, p_0)}))$$

is equal to

$$\widetilde{(\sigma^{\tilde{\Sigma}} \oplus \sigma^T)_{[[\gamma], t]_{\pi_1(\Sigma, p_0)}}}(T_{([\gamma], t)} p(X_{\tilde{\Sigma} \times T}([\gamma], t)), T_{([\gamma], t)} p(Y_{\tilde{\Sigma} \times T}([\gamma], t)))$$

which is equal to

$$\begin{aligned} (\sigma^{\tilde{\Sigma}} \oplus \sigma^T)_{([\gamma], t)}(X_{\tilde{\Sigma} \times T}([\gamma], t)), Y_{\tilde{\Sigma} \times T}([\gamma], t)) &= \sigma_t^T(X_T(t), Y_T(t)) \\ &= \sigma^t(X, Y). \end{aligned} \quad (3.14)$$

where in the first equality of (3.14) we have used that the vector fields $X_{\tilde{\Sigma} \times T}, Y_{\tilde{\Sigma} \times T}$ are tangent to the T -orbits $\{y\} \times T$ of $\tilde{\Sigma} \times T$, and the symplectic form vanishes on the orthogonal complements $T_{(u, t)}(\tilde{\Sigma} \times \{t\})$ to the T -orbits. The last equality follows from the definition of σ^t .

Finally let $\Omega_{\text{model}}^\Sigma$ stand for the flat connection on M_{model}^Σ given by the symplectic orthogonal complements to the tangent spaces to the orbits of the T -actions, see Proposition 2.17, and let $\mu_h^{\Omega_{\text{model}}^\Sigma}$ stand for the induced homomorphism $\mu_h^{\Omega_{\text{model}}^\Sigma}: H_1(M_{\text{model}}^\Sigma/T, \mathbb{Z}) \rightarrow T$ in homology. If $f: H_1(M_{\text{model}}^\Sigma/T, \mathbb{Z}) \rightarrow H_1(\Sigma, \mathbb{Z})$ is the group isomorphism induced by the symplectomorphism

$$M_{\text{model}}^\Sigma/T \rightarrow \tilde{\Sigma}/\pi_1(\Sigma, p_0) \rightarrow \Sigma, \quad (3.15)$$

where each arrow in (3.15) represents the natural map,

$$\mu_h^{\Omega_{\text{model}}^\Sigma} = \mu_h \circ f. \quad (3.16)$$

Because f is induced by a diffeomorphism, by Lemma 3.4 f preserves the intersection form and hence the unique collection of elements $\alpha'_i, \beta'_i, 1 \leq i \leq g$ such that $f(\alpha'_i) = \alpha_i$ and $f(\beta'_i) = \beta_i$, for

all $1 \leq i \leq g$, is a symplectic basis of the homology group $H_1(M_{\text{model}}^\Sigma/T, \mathbb{Z})$. Let $\hat{\gamma}$ be the $2g$ -tuple of elements $\mu_h^{\Omega_{\text{model}}^\Sigma}(\alpha'_i), \mu_h^{\Omega_{\text{model}}^\Sigma}(\beta'_i)$, $1 \leq i \leq g$. Therefore by (3.16)

$$\hat{\gamma} = (\mu_h(\alpha_i), \mu_h(\beta_i))_{i=1}^g.$$

The result follows because the $\text{Sp}(2g, \mathbb{Z})$ -orbit of $(\mu_h(\alpha_i), \mu_h(\beta_i))_{i=1}^g$ is equal to item 4) in Definition 3.9. \square

3.4 Classification Theorem

We state and prove the two main results of Section 3, by putting together previous results.

Theorem 3.15. *Let T be a $(2n - 2)$ -dimensional torus. Let (M, σ) be a compact connected $2n$ -dimensional symplectic manifold on which T acts freely and symplectically and such that at least one, and hence every T -orbit is a $(2n - 2)$ -dimensional symplectic submanifold of (M, σ) .*

Then the list of ingredients of (M, σ, T) as in Definition 3.9 is a complete set of invariants of (M, σ, T) , in the sense that, if (M', σ') is a compact connected $2n$ -dimensional symplectic manifold equipped with a free symplectic action of T for which at least one, and hence every T -orbit is $(2n - 2)$ -dimensional symplectic submanifold of (M', σ') , (M', σ') is T -equivariantly symplectomorphic to (M, σ) if and only if the list of ingredients of (M', σ', T) is equal to the list of ingredients of (M, σ, T) .

And given a list of ingredients for T , as in Definition 3.13, there exists a symplectic $2n$ -dimensional manifold (M, σ) with a free torus action of T for which one at least one, and hence every T -orbit is a $(2n - 2)$ -dimensional symplectic submanifold of (M, σ) , such that the list of ingredients of (M, σ, T) is equal to the list of ingredients for T .

Proof. It follows by putting together Lemma 3.11, Proposition 3.12 and Proposition 3.14. Observe that the combination of Lemma 3.11, Proposition 3.12 gives the existence part of the theorem, while Proposition 3.14 gives the existence part. \square

Corollary 3.16. *Let (M, σ) be a compact connected symplectic $2n$ -dimensional manifold equipped with a free symplectic action of $(2n - 2)$ -dimensional torus T for which at least one, and hence every T -orbit is a $(2n - 2)$ -dimensional symplectic submanifold of (M, σ) .*

Then the genus g of the surface M/T is a complete invariant of the T -equivariant diffeomorphism type of M , in the following sense. If (M', σ') is a compact, connected, $2n$ -dimensional symplectic manifold equipped with a free action of a $(2n - 2)$ -dimensional torus such that at least one, and hence every T -orbit is a $(2n - 2)$ -dimensional symplectic submanifold of (M', σ') , M' is T -equivariantly diffeomorphic to M if and only if the genus of the orbit space M'/T is equal to the genus of M/T .

Moreover, given any non-negative integer, there exists a symplectic $2n$ -dimensional manifold (M, σ) with at least one, and hence every T -orbit being a $(2n - 2)$ -dimensional symplectic submanifold of (M, σ) , such that the genus of M/T is precisely the aforementioned integer.

The proof of Corollary 3.16 is immediate. Indeed, by Corollary 2.40, M is T -equivariantly diffeomorphic to the Cartesian product $M/T \times T$ equipped with the action of T by translations

on the right factor of the product. Suppose that Φ is a T -equivariant symplectomorphism from (M, σ) to (M', σ') . Because Φ is T -equivariant, it descends to a diffeomorphism $\hat{\Phi}$ from the orbit space M/T onto the orbit space M'/T , and hence the genus of M/T equals the genus of M'/T . Conversely, suppose that (M, σ) and (M', σ') are such that the genus of M/T equals the genus of M'/T . By the classification theorem of compact, connected, orientable (boundaryless of course) surfaces, there exists a diffeomorphism $F: M/T \rightarrow M'/T$. Hence the map $M/T \times T \rightarrow M'/T \times T$ given by $(x, t) \rightarrow (F(x), t)$ is a T -equivariant diffeomorphism, and by Corollary 2.40 we are done. Now let g be a non-negative integer, and let Σ be a (compact, connected, orientable) surface of genus g , and let T be a $(2n - 2)$ -dimensional torus. Then $M_g := \Sigma \times T$ is a $2n$ -dimensional manifold. Equip it with any product symplectic form, and with the action of T by translations on the right factor. Then the T -orbits, which are of the form $\{u\} \times T$, $u \in \Sigma$, are symplectic submanifolds of M_g . Clearly M_g/T is diffeomorphic to Σ by means of $[u, t] \mapsto u$.

Remark 3.17 We summarize Corollary 3.16 in the language of categories, c.f. MacLane's book [32]. Let T be a torus and let \mathcal{M} denote the category of which the objects are the compact connected symplectic manifolds (M, σ) together with a free symplectic T -action on (M, σ) , such that at least one, and hence every T -orbit is a $(2n - 2)$ -dimensional symplectic submanifold of M , and of which the morphisms are the T -equivariant symplectomorphisms of (M, σ) . Let \mathcal{Z}^+ denote category which consists of the set of non-negative integers, and of which the identity is the only endomorphism of categories. Then the assignment $M \mapsto g$, where g is the genus of the surface M/T , is a full functor of categories from the category \mathcal{M} onto the category \mathcal{Z}^+ . In particular the proper class \mathcal{M}/\sim of isomorphism classes in \mathcal{M} is a set, and the functor $\iota: M \mapsto g$, where g is the genus of the surface M/T , induces a bijective mapping ι/\sim from the category \mathcal{M}/\sim onto the category \mathcal{Z}^+ .

Let \mathcal{I} denote the set of all lists of ingredients as in Definition 3.13, viewed as a category, and of which the identities are the only endomorphisms of categories. Then the assignment ι in Definition 3.9 is a full functor of categories from the category \mathcal{M} onto the category \mathcal{I} . In particular the proper class \mathcal{M}/\sim of isomorphism classes in \mathcal{M} is a set, and the functor $\iota: \mathcal{M} \rightarrow \mathcal{I}$ induces a bijective mapping ι/\sim from \mathcal{M}/\sim onto \mathcal{I} . The fact that the mapping $\iota: \mathcal{M} \rightarrow \mathcal{I}$ is a functor and the mapping ι/\sim is injective follows from the uniqueness part of Theorem 3.15, while the surjectivity of ι , follows from the existence part. \oslash

4 Classification

This section extends the results of Section 3 to non-free actions.

A tool needed to obtain the classification is Theorem 4.18, which is a characterization of geometric isomorphisms of orbifold homology, c.f. Definition 4.9. Such classification appears to be of independent interest as it generalizes a classical result about smooth surfaces to smooth orbisurfaces. Some of the statements and proofs in the present section are analogous to those of Section 3, and we do not repeat them.

4.1 Geometric Torsion in Homology of Orbifolds

Like for compact, connected, orientable smooth surfaces, there exists a classification for compact, connected, orientable smooth orbisurfaces.

Remark 4.1 It follows from Definition 2.9 that there are only finitely many points in compact, connected, smooth orientable orbifold \mathcal{O} which are singular, so the singular locus $\Sigma_{\mathcal{O}}$ is finite. \oslash

The compact, connected, boundaryless, orientable smooth orbisurfaces are classified by the genus of the underlying surface and the n -tuple of cone point orders $(o_k)_{k=1}^n$, where $o_i \leq o_{i+1}$, for all $1 \leq i \leq n - 1$, in the sense that the following two statements hold.

Theorem 4.2. *First, given a positive integer g and an n -tuple $(o_k)_{k=1}^n$, $o_i \leq o_{i+1}$ of positive integers, there exists a compact, connected, boundaryless, orientable smooth orbisurface orbifold \mathcal{O} with underlying topological space a compact, connected surface of genus g and n cone points of respective orders o_1, \dots, o_n . Secondly, let $\mathcal{O}, \mathcal{O}'$ be compact, connected, boundaryless, orientable smooth orbisurfaces. Then \mathcal{O} is diffeomorphic to \mathcal{O}' if and only if the genus of their underlying surface is the same, and their associated increasingly ordered n -tuple of orders of cone points are equal.*

Proof. If $\mathcal{O}, \mathcal{O}'$ are compact, connected, boundaryless, orientable smooth 2-dimensional orbifolds which are moreover diffeomorphic, it follows from the definition of diffeomorphism of orbifolds that the genus of their underlying surface is the same, and their associated increasingly ordered n -tuple of orders of cone points are equal.

Conversely suppose that $\mathcal{O}, \mathcal{O}'$ are boundaryless, orientable smooth 2-dimensional orbifolds with the property that the genus of their underlying surface is the same, and their associated increasingly ordered n -tuples of orders of cone points are equal. Write p_k for the cone points of \mathcal{O} ordered so that p_k has order o_k for all k , where $1 \leq k \leq n$. Because $\mathcal{O}, \mathcal{O}'$ are compact, connected, boundaryless, orientable and 2-dimensional, for each cone point in \mathcal{O} and each cone point in \mathcal{O}' there exists a neighborhood that is orbifold diffeomorphic to the standard model of the plane modulo a rotation. Therefore, since the tuples of orders of \mathcal{O} and \mathcal{O}' are the same, for each k there is a neighborhood D_k of p_k , a neighborhood D'_k of p'_k and a diffeomorphism $f_k: D_k \rightarrow D'_k$ such that $f(p_k) = p'_k$. By shrinking each D_k or D'_k if necessary, we may assume that the topological boundaries $\partial_{|\mathcal{O}|}(D_k), \partial_{|\mathcal{O}'|}(D'_k)$ are simple closed curves. The map $f: D := \bigcup D_k \rightarrow D' := \bigcup D'_k$ defined by $f|_{D_k} := f_k$ is an orbifold diffeomorphism such that $f(p_k) = p'_k$ and $f(D_k) = D'_k$ for all k , where $1 \leq k \leq n$. Since the topological boundaries $C := \partial_{|\mathcal{O}|}(D_k), C' := \partial_{|\mathcal{O}'|}(D'_k)$ are simple closed curves, $|\mathcal{O}| \setminus \bigcup D_k$ and $|\mathcal{O}'| \setminus \bigcup D'_k$ are surfaces with boundary, and their corresponding boundaries consist of precisely k boundary components, $\partial_{|\mathcal{O}|}(D_1), \dots, \partial_{|\mathcal{O}|}(D_n)$ and $\partial_{|\mathcal{O}'|}(D'_1), \dots, \partial_{|\mathcal{O}'|}(D'_n)$, each of which is a circle. Then by the classification of surfaces with boundary, there exists a diffeomorphism $g: C \rightarrow C'$. By definition of diffeomorphism, $g(\partial C) = \partial C'$, and hence there exists a permutation ρ of $\{1, \dots, n\}$ such that $g(\partial_{|\mathcal{O}|}(D_k)) = \partial_{|\mathcal{O}'|}(D'_{\rho(k)})$ for all k , where $1 \leq k \leq n$. There exist diffeomorphisms of a surface with boundary that permute the boundary circles in any way one wants, and hence by precomposing with an appropriate such diffeomorphism we may assume that τ is the identity. Because of this together with the fact that a diffeomorphism of a circle which preserves orientation is isotopic to the identity map, we can smoothly deform g near the

boundary in \mathcal{O} of each D_k so that g agrees with f on $\partial_{|\mathcal{O}|}(C)$. Hence the map $F: \mathcal{O} \rightarrow \mathcal{O}'$ given as $F|_C := g$ and $F|_{\mathcal{O} \setminus C} := f$ is a well defined orbifold diffeomorphism between \mathcal{O} and \mathcal{O}' . \square

Remark 4.3 A geometric classification of orbisurfaces which considers hyperbolic, elliptic and parabolic structures, is given by Thurston in [55, Th. 13.3.6] – while such statement is very interesting and complete, the most convenient classification statement for the purpose of this paper is only in differential topological terms, c.f Theorem 4.2. The author is grateful to A. Hatcher for providing him with the precise classification statement, and indicating to him how to prove it. \oslash

Definition 4.4 Let Σ be a smooth, compact, connected, orientable smooth orbisurface with n singular points. Fix an order in the singular points, say p_1, \dots, p_n , such that the order of p_k is less than or equal to the order of p_{k+1} . We say that a collection of n elements $\{\gamma_k\}_{k=1}^n \subset H_1^{\text{orb}}(\Sigma, \mathbb{Z})$ is a *geometric torsion basis*⁴ of $H_1^{\text{orb}}(\Sigma, \mathbb{Z})$ if γ_k is the homology class of a loop $\tilde{\gamma}_k$ obtained as an oriented boundary of a closed small disk containing the k^{th} singular point of the orbisurface Σ with respect to the aforementioned ordering, and where no two such disks intersect. \oslash

Definition 4.5 Let $\vec{o} = (o_k)_{k=1}^n$ be an n -tuple of integers. We call $S_n^{\vec{o}}$ the subgroup of the permutation group S_n which preserve the n -tuple \vec{o} , i.e. $S_n^{\vec{o}} := \{\tau \in S_n \mid (o_{\tau(k)})_{k=1}^n = \vec{o}\}$. \oslash

Lemma 4.6. *Let Σ be a compact, connected, orientable, boundaryless smooth orbisurface. Assume moreover that Σ is a good orbisurface. Then the order of any cone point of Σ is equal to the order of the homotopy class of a small loop around that point.*

Proof. Take a cone point with cone angle $2\pi/n$, and let γ denote the associated natural generator of $\pi_1^{\text{orb}}(\Sigma, x_0)$. We already know that $\gamma^n = 1$, because we have that relation in the presentation of the orbifold fundamental group. Since the orbifold has a manifold cover, the projection around the pre-image of the cone point is a n -fold branched cover, which implies that for any $k < n$, γ^k does not lift to the cover and so it must be nontrivial. \square

Let $g, n, o_k, 1 \leq k \leq n$, be non-negative integers, and let Σ be a compact, connected, orientable smooth orbisurface with underlying topological space a surface of genus g , and with n singular points p_k of respective orders o_k . The orbifold fundamental group $\pi_1^{\text{orb}}(\Sigma, p_0)$ has group presentation

$$\langle \{\alpha_i, \beta_i\}_{i=1}^g, \{\gamma_k\}_{k=1}^n \mid \prod_{k=1}^n \gamma_k = \prod_{i=1}^g [\alpha_i, \beta_i], \gamma_k^{o_k} = 1, 1 \leq k \leq n \rangle, \quad (4.1)$$

where the elements $\alpha_i, \beta_i, 1 \leq i \leq g$ form a symplectic basis of the free homology $H_1^{\text{orb}}(\Sigma, \mathbb{Z})_F$, and the γ_k form a geometric torsion basis of the orbifold homology group $H_1^{\text{orb}}(\Sigma, \mathbb{Z})$. By abelianizing expression (4.1) we obtain the first integral orbifold homology group $H_1^{\text{orb}}(\Sigma, \mathbb{Z})$

$$\langle \{\alpha_i, \beta_i\}_{i=1}^g, \{\gamma_k\}_{k=1}^n \mid \sum_{k=1}^n \gamma_k = 0, o_k \gamma_k = 0, 1 \leq k \leq n \rangle. \quad (4.2)$$

⁴We use the word “torsion” because a geometric torsion basis will generate the torsion subgroup of the orbifold homology group. Similarly, we use “geometric” because the homology classes come from geometric elements, loops around singular points.

The torsion subgroup $H_1^{\text{orb}}(\Sigma, \mathbb{Z})_T$ of the first orbifold integral homology group of the orbisurface Σ is generated by the geometric torsion $\{\gamma_k\}_{k=1}^n$ with the relations $o_k \gamma_k = 0$ and $\sum_{k=1}^n \gamma_k = 0$. There are many free subgroups F of the first orbifold homology group for which $H_1^{\text{orb}}(\Sigma, \mathbb{Z}) = F \oplus H_1^{\text{orb}}(\Sigma, \mathbb{Z})_T$. In what follows we will use the definition

$$H_1^{\text{orb}}(\Sigma, \mathbb{Z})_F := H_1^{\text{orb}}(\Sigma, \mathbb{Z}) / H_1^{\text{orb}}(\Sigma, \mathbb{Z})_T, \quad (4.3)$$

and we call the left hand side of expression (4.3), the *free first orbifold homology group of the orbisurface Σ* ; such quotient group is isomorphic to the free group on $2g$ generators \mathbb{Z}^{2g} , and there is an isomorphism of groups from $H_1^{\text{orb}}(\Sigma, \mathbb{Z})$ onto $H_1^{\text{orb}}(\Sigma, \mathbb{Z})_F \oplus H_1^{\text{orb}}(\Sigma, \mathbb{Z})_T$. As in (3.2), there is a natural *intersection form*

$$\cap_F : H_1^{\text{orb}}(\Sigma, \mathbb{R})_F \otimes H_1^{\text{orb}}(\Sigma, \mathbb{R})_F \rightarrow \mathbb{R}, \quad (4.4)$$

which for simplicity we write $\cap_F = \cap$, and a natural isomorphism

$$H_1^{\text{orb}}(\Sigma, \mathbb{R})_F \rightarrow H_1(\hat{\Sigma}, \mathbb{R}) \quad (4.5)$$

which pullbacks \cap to $\cap = \cap_F$, where $\hat{\Sigma}$ denotes the underlying surface to Σ .

Example 4.7 Let Σ be a compact, connected, orientable smooth orbisurface with underlying topological space equal to a 2-dimensional torus $(\mathbb{R}/\mathbb{Z})^2$. Suppose that Σ has precisely one cone point p_1 of order $o_1 = 2$. Let $\tilde{\gamma}$ be obtained as boundary loop of closed small disk containing the singular point p_1 , and let $\gamma = [\tilde{\gamma}]$. Let α, β be representative of the standard basis of loops of the surface underlying Σ (recall that α, β are a basis of the quotient free first orbifold homology group of Σ). Then for any point $x_0 \in \Sigma$

$$\pi_1^{\text{orb}}(\Sigma, x_0) = \langle \alpha, \beta, \gamma \mid [\alpha, \beta] = \gamma, \gamma^2 = 1 \rangle,$$

and

$$H_1^{\text{orb}}(\Sigma, \mathbb{Z}) = \langle \alpha, \beta, \gamma \mid \gamma = 1, 2\gamma = 0 \rangle \simeq \langle \alpha, \beta \rangle.$$

∅

Remark 4.8 If (M, σ) is a compact connected symplectic manifold, T is a torus which acts effectively but non-freely on it by symplectomorphisms and with $\dim T$ -dimensional symplectic T -orbits, there exists at least one non-trivial element $[\gamma] \in H_1^{\text{orb}}(M/T, \mathbb{Z})$ for which there is an integer $k \in \mathbb{Z}^+$ such that $k[\gamma] = 0$ and $\mu_h([\gamma]) \neq 0$. (c.f. Theorem 2.41 for a global result). ∅

4.2 Monodromy invariant

We define the Fuchsian signature monodromy space, whose elements give one of the ingredients of the classification theorems.

4.2.1 Geometric Isomorphisms

An orbifold diffeomorphism between orbisurfaces induces an isomorphism at the level of orbifold fundamental groups, and at the level of first orbifold homology groups.

Definition 4.9 Let $\mathcal{O}, \mathcal{O}'$ be compact, connected, orientable smooth orbisurfaces. We say that an isomorphism $H_1^{\text{orb}}(\mathcal{O}, \mathbb{Z}) \rightarrow H_1^{\text{orb}}(\mathcal{O}', \mathbb{Z})$ is *orbisurface geometric* if there exists an orbifold diffeomorphism $\mathcal{O} \rightarrow \mathcal{O}'$ which induces it. \square

Example 4.10 If (M, σ) is a compact and connected symplectic manifold of dimension $2n$, and if T is a $(2n - 2)$ -dimensional torus which acts freely on (M, σ) by means of symplectomorphisms and whose T -orbits are $(2n - 2)$ -dimensional symplectic submanifold of (M, σ) , then the torsion part of the first integral orbifold homology group of the surface M/T is trivial. But if the action of the torus is not free then M/T is an orbisurface, and the torsion subgroup is frequently non-trivial, although in a few cases it is trivial. For example:

- a) if there is only one singular point in M/T , or
- b) if there are precisely two cone points of orders 2 and 3 in M/T , or in general of orders k and $k + 1$, for a positive integer k . In this case it is easy to see that the torsion subgroup of the first integral orbifold homology group is trivial because it is generated by γ_1, γ_2 with the relations $k\gamma_1 = (k + 1)\gamma_2 = 0$, and $\gamma_1 + \gamma_2 = 0$.
- c) if there are precisely three cone points of orders 3, 4, 5 in M/T , or more generally any three points whose orders are coprime.

However, in cases a) and b) the orbifolds are not good, so they do not arise as M/T , c.f. Lemma 2.34. On the other hand, in case c) the orbifold is good, and as we will prove later it does arise as the orbit space of many symplectic manifolds (M, σ) . In this case it is possible to give a description of the monodromy invariant which is analogous to the free case done in Section 3. \square

Recall that, by Lemma 3.4, if Σ, Σ' are compact, connected, oriented, smooth orbisurfaces of the same Fuchsian signature, and if $f: H_1^{\text{orb}}(\Sigma, \mathbb{Z}) \rightarrow H_1^{\text{orb}}(\Sigma', \mathbb{Z})$ is a group isomorphism for which there exists an orientation preserving orbifold diffeomorphism $i: \Sigma \rightarrow \Sigma'$ such that $f = i_*$, then f preserves the intersection form.

Example 4.11 Let \mathcal{O} be any orbifold with two cone points of orders 10 and 15. The torsion part of the orbifold homology is isomorphic to the quotient of the additive group $\mathbb{Z}_{10} \oplus \mathbb{Z}_{15}$ by the sum of the two obvious generators. This group is isomorphic to \mathbb{Z}_5 . Take an isomorphism of \mathbb{Z}_5 , which squares each element. This isomorphism cannot be realized by an orbifold diffeomorphism. This is the case because an orbifold diffeomorphism has to be multiplication by 1 or -1 on each of \mathbb{Z}_{10} and \mathbb{Z}_{15} , so it has to be multiplication by 1 or -1 on the quotient \mathbb{Z}_5 . \square

Not every automorphism of the first orbifold homology group preserves the order of the orbifold singularities.

Example 4.12 Let \mathcal{O} be a compact, connected, orientable smooth orbisurface with underlying topological space equal to a 2-dimensional torus. Suppose that \mathcal{O} has precisely two cone points

p_1, p_2 of respective orders $o_1 = 5$ and $o_2 = 10$. Then for any point $x_0 \in \mathcal{O}$

$$\pi_1^{\text{orb}}(\mathcal{O}, x_0) = \langle \alpha, \beta, \gamma_1, \gamma_2 \mid \gamma_1 \gamma_2 = [\alpha, \beta], \gamma_1^5 = \gamma_2^{10} = 1 \rangle,$$

and

$$H_1^{\text{orb}}(\mathcal{O}, \mathbb{Z}) = \langle \alpha, \beta, \gamma_1, \gamma_2 \mid \gamma_1 + \gamma_2 = 0, 5\gamma_1 = 10\gamma_2 = 0 \rangle.$$

The assignment

$$F: \alpha \rightarrow \alpha, \beta \rightarrow \beta, \gamma_1 \mapsto \gamma_2, \gamma_2 \mapsto \gamma_1 \quad (4.6)$$

defines a group automorphism of $H_1^{\text{orb}}(\mathcal{O}, \mathbb{Z})$. If this automorphism is induced by a diffeomorphism, then the same map on $\pi_1^{\text{orb}}(\mathcal{O}, x_0)$ should be an isomorphism, which is false since in $\pi_1^{\text{orb}}(\mathcal{O}, x_0)$ the classes γ_1 and γ_2 have different orders. Thus the assignment (4.6) is not geometric. \ominus

4.2.2 Symplectic and torsion geometric maps

Next we introduce the notion of symplectic isomorphism as well as that of singularity–order preserving isomorphism.

Let K, L be arbitrary groups, and let $h: K \rightarrow L$ be a group isomorphism. We denote by K_T, L_T the corresponding torsion subgroups, by K_F, L_F the quotients $K/K_T, L/L_T$, by h_T the restriction of h to $K_T \rightarrow L_T$, and by h_F the restriction of h to $K_F \rightarrow L_F$.

Definition 4.13 Let $\mathcal{O}, \mathcal{O}'$ be compact, connected, orientable smooth orbisurfaces. We say that an isomorphism $Z = H_1^{\text{orb}}(\mathcal{O}, \mathbb{Z}) \rightarrow Z' = H_1^{\text{orb}}(\mathcal{O}', \mathbb{Z})$ is *torsion geometric* if and only if the isomorphism $Z_T \rightarrow Z'_T$ sends a geometric torsion basis to a geometric torsion basis preserving the order of the orbifold singularities. \ominus

Definition 4.14 Let $\mathcal{O}, \mathcal{O}'$ be compact, connected, orientable smooth orbisurfaces. We say that an isomorphism $Z = H_1^{\text{orb}}(\mathcal{O}, \mathbb{Z}) \rightarrow Z' = H_1^{\text{orb}}(\mathcal{O}', \mathbb{Z})$ is *symplectic* if the isomorphism $Z_F \rightarrow Z'_F$ respects the symplectic form, c.f. (4.4), i.e. the matrix of the isomorphism w.r.t. symplectic bases is in the integer symplectic group. \ominus

Definition 4.15 Let Z, Z' be respectively the first integral orbifold homology groups of compact connected orbisurfaces $\mathcal{O}, \mathcal{O}'$ of the same Fuchsian signature. We define the following set of isomorphisms

$$S^{Z, Z'} := \{f \in \text{Iso}(Z, Z') \mid f \text{ is torsion geometric}\}, \quad (4.7)$$

and we denote by

$$\text{Sp}(Z, Z') := \{h \in \text{Iso}(Z, Z') \mid h \text{ is symplectic}\}. \quad (4.8)$$

\ominus

Definition 4.16 Let Z_1, Z_2 be respectively the first integral orbifold homology groups of compact connected orbisurfaces $\mathcal{O}_1, \mathcal{O}_2$ of the same Fuchsian signature. Let $f_i: Z_i \rightarrow T$ be homomorphisms into a torus T . We say that f_1 is $\text{Sp}(Z_1, Z_2) \cap S^{Z_1, Z_2}$ -equivalent to f_2 if there exists an isomorphism $i: Z_1 \rightarrow Z_2$ such that there is an identity of maps $f_2 = f_1 \circ i$ and $i \in \text{Sp}(Z_1, Z_2) \cap S^{Z_1, Z_2}$. \square

4.2.3 Geometric isomorphisms: characterization

The main result of this section is Theorem 4.18, where we give a characterization of geometric isomorphisms in terms of symplectic maps, c.f. Definition 4.14, and torsion geometric maps, c.f. Definition 4.13. Then we deduce from it Proposition 4.19 which is a key ingredient of the proof of the classification theorem.

Lemma 4.17. *Let $\Sigma_i, i = 1, 2$, be compact, connected, oriented smooth orbisurfaces with the same Fuchsian signature $(g; \vec{o})$, and suppose that G is an isomorphism from the first integral orbifold homology group $H_1^{\text{orb}}(\Sigma_1, \mathbb{Z})$ onto $H_1^{\text{orb}}(\Sigma_2, \mathbb{Z})$ which is symplectic and torsion geometric, c.f. Definition 4.14 and Definition 4.13. Then there exists an orbifold diffeomorphism $g: \Sigma_1 \rightarrow \Sigma_2$ such that $G_T = (g_*)_T$ and $G_F = (g_*)_F$.*

Proof. Because an orbifold is classified up to orbifold diffeomorphisms by its Fuchsian signature, c.f. Theorem 4.2, without loss of generality we may assume that $\Sigma = \Sigma_1 = \Sigma_2$, and let $\{\gamma_k\}$ be a geometric torsion basis of $H_1^{\text{orb}}(\Sigma, \mathbb{Z})$, c.f Definition 4.4. Choose an orbifold atlas for Σ such that the orbifold chart U_k around the k^{th} singular point p_k is homeomorphic to a disk D_k modulo a finite group of diffeomorphisms, and such that every singular point is contained in precisely one chart. The oriented boundary loop ∂U_k represents the class $\gamma_k \in H_1^{\text{orb}}(\Sigma, \mathbb{Z})_T$. This in particular implies that there exists an orbifold diffeomorphism $f_\tau: U := \bigcup_k U_k \rightarrow U$ such that $f_\tau(p_k) = p_{\tau(k)}$ and

$$f_\tau(U_k) = U_{\tau(k)}. \quad (4.9)$$

Replace each orbifold chart around a singular point of Σ by a manifold chart. This gives rise to a manifold atlas, which defines a compact, connected, smooth orientable surface $\widehat{\Sigma}$ without boundary⁵ with the same underlying space as that of the orbisurface Σ . Let $H_1^{\text{orb}}(\Sigma, \mathbb{Z})_F \rightarrow H_1(\widehat{\Sigma}, \mathbb{Z})$ be the natural intersection form preserving automorphism in (4.5). The automorphism obtained from G_F by conjugation with the aforementioned automorphism preserves the intersection form on $H_1(\widehat{\Sigma}, \mathbb{Z})$. By Lemma 3.4 there is a surface diffeomorphism which induces it, which sends U_k to $U_{\tau(k)}$. Because each diffeomorphism of a circle is isotopic to a rotation or a reflection, by (4.9) the restriction of the aforementioned surface diffeomorphism to the punctured surface $\widehat{\Sigma} \setminus \bigcup_k U_k$ may be glued to f_τ , along the boundary circles ∂U_k , to give rise to an orbifold diffeomorphism of Σ which satisfies the required properties. \square

Theorem 4.18. *Let Σ_1, Σ_2 be compact, connected, orientable smooth orbisurfaces of the same Fuchsian signature. An isomorphism $Z_1 = H_1^{\text{orb}}(\Sigma_1, \mathbb{Z}) \rightarrow Z_2 = H_1^{\text{orb}}(\Sigma_2, \mathbb{Z})$ is orbisurface geometric if and only if it is symplectic and torsion geometric.*

⁵recall that we are always assuming, unless otherwise specified, that all manifolds and orbifolds in this paper have no boundary.

Proof. Suppose that $G: Z_1 \rightarrow Z_2$ is an orbisurface geometric isomorphism, c.f. Definition 4.9. It follows from the definition of orbifold diffeomorphism that if the group isomorphism G is induced by an orbifold diffeomorphism, then the induced map G_F on the free quotient preserves the intersection form and G sends geometric torsion basis to geometric torsion basis of the orbifold homology preserving the order of the orbifold singularities.

Conversely, suppose that G is symplectic and torsion geometric, c.f. Definition 4.14, Definition 4.13. By Theorem 4.2 we may assume without loss of generality that $\Sigma = \Sigma_1 = \Sigma_2$, so G is an automorphism of $H_1^{\text{orb}}(\Sigma, \mathbb{Z})$. By Lemma 4.17 there exists an orbifold diffeomorphism $g: \Sigma \rightarrow \Sigma$ such that

$$(g_*)_F = G_F, \quad (g_*)_T = G_T. \quad (4.10)$$

Let us define the mapping

$$K := g_* \circ G^{-1}: H_1^{\text{orb}}(\Sigma, \mathbb{Z}) \rightarrow H_1^{\text{orb}}(\Sigma, \mathbb{Z}). \quad (4.11)$$

K given by (4.11) is a group isomorphism because it is the composite of two group isomorphisms. Because of the identities in expression (4.10), K satisfies that

$$K_T = \text{Id}_{H_1^{\text{orb}}(\Sigma, \mathbb{Z})_T}, \quad K_F = \text{Id}_{H_1^{\text{orb}}(\Sigma, \mathbb{Z})_F}. \quad (4.12)$$

If the genus of the underlying surface $|\Sigma|$ is 0, then $H_1^{\text{orb}}(\Sigma, \mathbb{Z})_F$ is trivial and $H_1^{\text{orb}}(\Sigma, \mathbb{Z})_T = H_1^{\text{orb}}(\Sigma, \mathbb{Z})$, $K = K_T = \text{Id}$, so $g_* = G$, and we are done. If the genus of $|\Sigma|$ is strictly positive, then there are two cases. First of all, if K is the identity map, then $g_* = G$, and we are done. If otherwise K is not the identity map, then $g_* \neq G$, and let us choose a free subgroup F of $H_1^{\text{orb}}(\Sigma, \mathbb{Z})$ such that $H_1^{\text{orb}}(\Sigma, \mathbb{Z}) = F \oplus H_1^{\text{orb}}(\Sigma, \mathbb{Z})_T$, and a symplectic basis α_i, β_i of the free group F . To make the forthcoming notation simpler rename $e_{2i-1} = \alpha_i$ and $e_{2i} = \beta_i$, for all i such that $1 \leq i \leq g$. By the right hand side of equation (4.12), we have that there exist non-negative integers a_k^i such that if n is the number singular of points of Σ , then

$$K(e_i) = e_i + \sum_{k=1}^n a_k^i \gamma_k. \quad (4.13)$$

By the left hand side of (4.12),

$$K(\gamma_k) = \gamma_k. \quad (4.14)$$

Next we define a new isomorphism, which we call K_{new} , by altering K in the following fashion.

Choose an orbifold atlas for the orbisurface Σ such that for each singular point p_j of Σ there is a unique orbifold chart which contains it, and which is homeomorphic to a disk D_j modulo a finite group of diffeomorphisms. Let $\hat{\Sigma}$ be the smooth surface whose underlying topological space is the underlying surface $|\Sigma|$ to Σ , and whose smooth structure is given by the atlas defined by replacing each chart which contains a singular point by a manifold chart from the corresponding disk. Let \tilde{e}_i be a loop which represents the homology class $e_i \in H_1^{\text{orb}}(\Sigma, \mathbb{Z})$. Take an annulus \mathcal{A} in Σ such that the loop \tilde{e}_i crosses \mathcal{A} exactly once, in the sense that it intersects the boundary $\partial\mathcal{A}$

of \mathcal{A} twice, once at each of the two boundary components of $\partial\mathcal{A}$, and such that it contains the j^{th} singular point p_j of Σ which is enclosed by the oriented loop whose homology class is γ_j , and no other singular point of Σ . This can be done by choosing the annulus \mathcal{A} in such a way that its boundary curves represent the same class as the dual element e_{i+1} to e_i (instead of e_{i+1} we may have e_{i-1} , depending on how the symplectic basis of e_i is arranged). Equip the topological space $|\mathcal{A}| \subset |\Sigma| = |\widehat{\Sigma}|$ with the smooth structure which comes from restricting to $|\mathcal{A}|$ the charts of the smooth structure of $\widehat{\Sigma}$, and let $\widehat{\mathcal{A}}$ be the smooth submanifold-with-boundary of $\widehat{\Sigma}$ which arises in such way. The loop \tilde{e}_i intersects the boundary of the annulus at an initial point x_{ij} and at an end point y_{ij} , and intersects the annulus itself at a curved segment path $[x_{ij}, y_{ij}]$, which by possibly choosing a different representative of the class e_i , may be assumed to be a smooth embedded 1-dimensional submanifold-with-boundary of $\widehat{\mathcal{A}}$. Replace the segment $[x_{ij}, y_{ij}]$ of the loop \tilde{e}_i by a smoothly embedded path $P(x_{ij}, y_{ij})$ which starts at the point x_{ij} , goes towards the cone point enclosed by γ_j while inside of the aforementioned annulus, and goes around it precisely once, to finally come back to end up at the end point y_{ij} . The replacement of the segment $[x_{ij}, y_{ij}]$ by the path $P(x_{ij}, y_{ij})$ gives rise to a new loop \tilde{e}'_i , which agrees with the loop \tilde{e}_i outside of \mathcal{A} .

We claim that there exists an orbifold diffeomorphism $\widehat{h}_{ij}: \widehat{\Sigma} \rightarrow \widehat{\Sigma}$ which is the identity map outside of the annulus and which sends the segment $[x_{ij}, y_{ij}]$ to the path $P(x_{ij}, y_{ij})$, and hence the loop \tilde{e}_i to the loop \tilde{e}'_i . Indeed, let \widehat{f} be a diffeomorphism of the aforementioned annulus $\widehat{\mathcal{A}}$ with

$$\widehat{f}([x_{ij}, y_{ij}]) = P(x_{ij}, y_{ij}),$$

and with \widehat{f} being the identity on $\partial\widehat{\mathcal{A}}$. Then let $\widehat{k}: \widehat{\mathcal{A}} \rightarrow \widehat{\mathcal{A}}$ be a diffeomorphism which is the identity on $P(x_{ij}, y_{ij})$ and on $\partial\widehat{\mathcal{A}}$, such that $\widehat{k}(\widehat{f}(p_j)) = p_j$. Such a diffeomorphism \widehat{k} exists since cutting the annulus along $P(x_{ij}, y_{ij})$ gives a disk, and in a disk there is a diffeomorphism taking any interior point x to any other interior point y and fixing a neighborhood of the boundary. The composition $\widehat{h}'_{ij} := \widehat{k} \circ \widehat{f}: \widehat{\mathcal{A}} \rightarrow \widehat{\mathcal{A}}$ is a diffeomorphism which is the identity on $\partial\widehat{\mathcal{A}}$, which sends the segment $[x_{ij}, y_{ij}]$ to the path $P(x_{ij}, y_{ij})$ and such that

$$\widehat{h}'_{ij}(p_j) = p_j. \tag{4.15}$$

Because the mapping \widehat{h}'_{ij} is the identity along $\partial\widehat{\mathcal{A}}$, it extends to a diffeomorphism \widehat{h}_{ij} along such boundary, which satisfies the required properties. Since p_j is contained in a unique orbifold chart and (4.15) holds, the way in which we defined the smooth structure of $\widehat{\Sigma}$ from the orbifold structure of Σ gives that \widehat{h}_{ij} defines an orbifold diffeomorphism $\Sigma \rightarrow \Sigma$. To emphasize that \widehat{h}_{ij} is a diffeomorphism at the level of orbifolds, we denote it by $h_{ij}: \Sigma \rightarrow \Sigma$.

The isomorphism h_{ij}^* induced on the orbifold homology by the orbifold diffeomorphism h_{ij} is given by

$$h_{ij}^*(e_k) = e_k, \quad k \neq i, \quad h_{ij}^*(e_i) = e_i + \gamma_j, \tag{4.16}$$

$$h_{ij}^*(\gamma_k) = \gamma_k, \tag{4.17}$$

where $1 \leq i \leq 2g$ and $1 \leq k \leq n$, and notice that in the first equality we have used that the boundary curve of the annulus is in the class of e_{i+1} . Define $f_{ij}: \Sigma \rightarrow \Sigma$ to be the orbifold

diffeomorphism obtained by composing h_{ij} with itself precisely a_i^j times. It follows from (4.16) and (4.17) that

$$f_{ij}^*(e_k) = e_k \text{ if } k \neq i, \quad f_{ij}^*(e_i) = e_i + a_i^j \gamma_j, \quad (4.18)$$

and

$$f_{ij}^*(\gamma_k) = \gamma_k, \quad (4.19)$$

where $1 \leq i \leq 2g$ and $1 \leq k \leq n$.

The isomorphisms f_{ij}^* commute with each other, because they are the identity on the torsion subgroup, and only change one loop of the free part which does not affect the other loops⁶. Therefore combining expressions (4.13), (4.18) and (4.19) we arrive at the identity

$$((f_{ij}^*)^{-1} \circ K)(e_i) = e_i + \sum_{k=1, k \neq j}^n a_k^i \gamma_k. \quad (4.20)$$

On the other hand, it follows from (4.14) and (4.19) that

$$((f_{ij}^*)^{-1} \circ K)(\gamma_k) = \gamma_k. \quad (4.21)$$

Now consider the isomorphism K_{new} of the orbifold homology group defined by

$$K_{\text{new}} := (\bigcirc_{1 \leq i \leq 2g, 1 \leq j \leq n} (f_{ij}^*)^{-1}) \circ K, \quad (4.22)$$

where recall that in (4.22), g is the genus of the surface underlying Σ , and n is the number of singular points. It follows from expression (4.21), and from (4.20), by induction on i and j , that K_{new} given by (4.22) is the identity map on the orbifold homology, which then by formula (4.11) implies that

$$\begin{aligned} G = g_* \circ K^{-1} &= g_* \circ (\bigcirc_{1 \leq i \leq 2g, 1 \leq j \leq n} f_{ij}^*) \\ &= (g \circ (\bigcirc_{1 \leq i \leq 2g, 1 \leq j \leq n} f_{ij}))_*, \end{aligned}$$

and hence G is a geometric isomorphism induced by

$$g \circ (\bigcirc_{1 \leq i \leq 2g, 1 \leq j \leq n} f_{ij}),$$

which is a composite of orbifold diffeomorphisms, and hence an orbifold diffeomorphism itself. \square

Recall that the mappings μ_h , μ'_h are, respectively, the homomorphisms induced on homology by the monodromy homomorphisms μ , μ' of the connections Ω , Ω' of symplectically orthogonal complements to the tangent spaces to the T -orbits in M , M' , respectively (c.f. Proposition 2.17). Recall that ν , ν' are the unique 2-forms respectively on M/T and M'/T such that $\pi^*\nu|_{\Omega_x} = \sigma|_{\Omega_x}$ and $\pi'^*\nu'|_{\Omega'_{x'}} = \sigma'|_{\Omega'_{x'}}$, for every $x \in M$, $x' \in M'$.

⁶Observe that this is completely false in the fundamental group.

Proposition 4.19. *Let $(M, \sigma), (M', \sigma')$ be compact, connected $2n$ -dimensional symplectic manifolds equipped with an effective symplectic action of a $(2n - 2)$ -dimensional torus T for which at least one, and hence every T -orbit is a $(2n - 2)$ -dimensional symplectic submanifold of (M, σ) and (M', σ') , respectively. Let $K = H_1^{\text{orb}}(M/T, \mathbb{Z})$ and similarly K' . Suppose that the symplectic orbit spaces $(M/T, \nu)$ and $(M'/T, \nu')$ are orbifold symplectomorphic and that μ_h is $\text{Sp}(K', K) \cap S^{K', K}$ -equivalent to μ'_h via an automorphism G from the first integral orbifold homology group $H_1^{\text{orb}}(M'/T, \mathbb{Z})$ onto $H_1^{\text{orb}}(M/T, \mathbb{Z})$ (c.f. Corollary 2.28 and Lemma 2.29 for the definitions of ν, ν' , or see the preceding paragraphs). Then there exists an orbifold diffeomorphism $g: M'/T \rightarrow M/T$ such that $G = g_*$ and $\mu'_h = \mu_h \circ g_*$.*

Proof. It follows from Theorem 4.18 applied to the groups Z', Z , which respectively are the first integral orbifold homology group of Σ' , and of Σ , where $\Sigma' = M'/T, \Sigma = M/T$. \square

4.2.4 Fuchsian signature space

We define the invariant of (M, σ) which encodes the monodromy of the connection for $\pi: M \rightarrow M/T$ of symplectic orthogonal complements to the tangent spaces to the T -orbits, c.f. Definition 4.24.

Definition 4.20 Let \mathcal{O} be a smooth orbisurface with n cone points $p_k, 1 \leq k \leq n$. The *Fuchsian signature* $\text{sig}(\mathcal{O})$ of \mathcal{O} is the $(n + 1)$ -tuple $(g; \vec{o})$ where g is the genus of the underlying surface to the orbisurface \mathcal{O} , o_k is the order of the point p_k , which we require to be strictly positive, and $\vec{o} = (o_k)_{k=1}^n$, where $o_k \leq o_{k+1}$, for all $1 \leq k \leq n - 1$. \oslash

Definition 4.21 Let \vec{o} be an n -dimensional tuple of strictly positive integers. We define

$$\mathcal{MS}_n^{\vec{o}} := \{B \in \text{GL}(n, \mathbb{Z}) \mid B \cdot \vec{o} = \vec{o}\},$$

i.e. $\mathcal{MS}_n^{\vec{o}}$ is the group of n -dimensional matrices which permute elements preserving the tuple of orders \vec{o} . \oslash

Definition 4.22 Let $(g; \vec{o})$ be an $(n + 1)$ -tuple of integers, where the o_k 's are strictly positive and non-decreasingly ordered. Let T be a torus. The *Fuchsian signature space associated to $(g; \vec{o})$* is the quotient space

$$T_{(g; \vec{o})}^{2g+n} / \mathcal{G}_{(g, \vec{o})} \quad (4.23)$$

where

$$T_{(g; \vec{o})}^{2g+n} := \{(t_i)_{i=1}^{2g+n} \in T^{2g+n} \mid \text{the order of } t_i \text{ is a multiple of } o_i, 2g + 1 \leq i \leq n\}.$$

and where $\mathcal{G}_{(g, \vec{o})}$ is the group of matrices

$$\mathcal{G}_{(g; \vec{o})} := \left\{ \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \in \text{GL}(2g + n, \mathbb{Z}) \mid A \in \text{Sp}(2g, \mathbb{Z}), D \in \mathcal{MS}_n^{\vec{o}} \right\}, \quad (4.24)$$

where $\mathrm{Sp}(2g, \mathbb{Z})$ is the group of $2g$ -dimensional symplectic matrices with integer entries, c.f. Section 3.1.1. and Definition 3.5, and $\mathcal{MS}_n^{\vec{o}}$ is the group of n -dimensional matrices which permute elements preserving the tuple of orders \vec{o} , c.f. Definition 4.21. \circledcirc

Remark 4.23 The lower left block C in the definition of $\mathcal{G}_{(g; \vec{o})}$ in Definition 4.22 of the description of item 4) is allowed to be any matrix. Intuitively, this reflects that there are many free subgroups of the first integral orbifold homology group with together with the torsion span the entire group. \circledcirc

Definition 4.24 Let (M, σ) be a smooth compact and connected symplectic manifold of dimension $2n$ and let T be a $(2n - 2)$ -dimensional torus which acts effectively on (M, σ) by means of symplectomorphisms. We furthermore assume that at least one, and hence every T -orbit is $(2n - 2)$ -dimensional symplectic submanifold of (M, σ) . Let $(g; \vec{o}) \in \mathbb{Z}^{1+m}$ be the Fuchsian signature of the orbit space M/T . Let $\{\gamma_k\}_{k=1}^m$ be a geometric torsion basis, c.f. Definition 4.4. Let $\{\alpha_i, \beta_i\}_{i=1}^g$ be a symplectic basis of a free subgroup of our choice of $H_1^{\mathrm{orb}}(M/T, \mathbb{Z})$, c.f. expression (3.3) and [37, Th. 2.3] whose direct sum with the torsion subgroup is equal to $H_1^{\mathrm{orb}}(M/T, \mathbb{Z})$. Let μ_h be the homomorphism induced on homology by the monodromy homomorphism μ associated to the connection Ω , c.f. Proposition 2.17. The *monodromy invariant* of (M, σ, T) is the $\mathcal{G}_{(g; \vec{o})}$ -orbit

$$\mathcal{G}_{(g; \vec{o})} \cdot ((\mu_h(\alpha_i), \mu_h(\beta_i))_{i=1}^g, (\mu_h(\gamma_k))_{k=1}^m), \quad (4.25)$$

of the $(2g+n)$ -tuple $((\mu_h(\alpha_i), \mu_h(\beta_i))_{i=1}^g, (\mu_h(\gamma_k))_{k=1}^m)$, where $\mathcal{G}_{(g; \vec{o})}$ is the group of matrices given in (4.24). \circledcirc

Because the invariant in Definition 4.24 depends on choices, it is unclear whether it is well defined.

Lemma 4.25. *Let (M, σ) be a smooth compact and connected symplectic manifold of dimension $2n$ and let T be a $(2n - 2)$ -dimensional torus which acts effectively on (M, σ) by means of symplectomorphisms. We furthermore assume that at least one, and hence every T -orbit is $(2n - 2)$ -dimensional symplectic submanifold of (M, σ) . The monodromy invariant of (M, σ, T) is well defined, in the following sense.*

Suppose that the signature of M/T is $(g; \vec{o}) \in \mathbb{Z}^{1+m}$. Let F, F' be any two free subgroups of $H_1^{\mathrm{orb}}(M/T, \mathbb{Z})$ whose direct sum with the torsion subgroup is equal to $H_1^{\mathrm{orb}}(M/T, \mathbb{Z})$, and let α_i, β_i and α'_i, β'_i be symplectic bases of F, F' , respectively. Let $\tau \in S_m^{\vec{o}}$. Let μ_h be the homomorphism induced in homology by means of the Hurewicz map from the monodromy homomorphism μ of the connection of symplectic orthogonal complements to the tangent spaces to the T -orbits, c.f. Proposition 2.17. Then the tuples $((\mu_h(\alpha_i), \mu_h(\beta_i))_{i=1}^g, (\mu_h(\gamma_k))_{k=1}^m)$ and $((\mu_h(\alpha'_i), \mu_h(\beta'_i))_{i=1}^g, (\mu_h(\gamma_{\tau(k)}))_{k=1}^m)$ lie in the same $\mathcal{G}_{(g; \vec{o})}$ -orbit in the Fuchsian signature space.

Proof. A symplectic basis of the maximal free subgroup F of $H_1^{\mathrm{orb}}(M/T, \mathbb{Z})$ may be taken to a symplectic basis of the maximal free subgroup F' of $H_1^{\mathrm{orb}}(M/T, \mathbb{Z})$ by a matrix

$$X := \begin{pmatrix} A & 0 \\ C & \mathrm{Id} \end{pmatrix} \in \mathrm{GL}(2g + m, \mathbb{Z}),$$

where the upper block A is a $2g$ -dimensional matrix in the integer symplectic linear group $\mathrm{Sp}(2g, \mathbb{Z})$, and the lower block C is $(m \times 2g)$ -dimensional matrix with integer entries. Here Id denotes the n -dimensional identity matrix, and 0 is the $(m \times 2g)$ -dimensional matrix all the entries of which equal 0 . A geometric torsion basis can be taken to another geometric torsion basis by preserving the order \vec{o} of the orbifold singularities by a matrix of the form

$$Y := \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & N \end{pmatrix} \in \mathrm{GL}(2g + m, \mathbb{Z}),$$

for a certain matrix $N \in \mathcal{MS}_m^{\vec{o}}$, and the product matrix

$$X Y = \begin{pmatrix} A & 0 \\ C & N \end{pmatrix}$$

lies in $\mathcal{G}_{(g; \vec{o})}$. □

With this matrix terminology, we can restate Proposition 4.19 in the following terms.

See the paragraph preceding Proposition 4.19 for a reminder of the terminology which we use next.

Proposition 4.26. *Let $(M, \sigma), (M', \sigma')$ be compact, connected $2n$ -dimensional symplectic manifolds equipped with an effective symplectic action of a $(2n - 2)$ -dimensional torus T for which at least one, and hence every T -orbit is a $(2n - 2)$ -dimensional symplectic submanifold of (M, σ) and (M', σ') , respectively. Suppose that the symplectic orbit spaces $(M/T, \nu)$ and $(M'/T, \nu')$ have Fuchsian signature $(g; \vec{o})$, that they are orbifold symplectomorphic and that the $\mathcal{G}_{(g; \vec{o})}$ -orbits of the $(2g + m)$ -tuples $((\mu_h(\alpha'_i), \mu_h(\beta'_i))_{i=1}^g, (\mu_h(\gamma'_k))_{k=1}^m)$ and $((\mu'_h(\alpha'_i), \mu'_h(\beta'_i))_{i=1}^g, (\mu'_h(\gamma'_k))_{k=1}^m)$ are equal, where α_i, β_i and α'_i, β'_i are respectively symplectic bases of free homology subgroups of $H_1^{\mathrm{orb}}(M/T, \mathbb{Z})$ and $H_1^{\mathrm{orb}}(M'/T, \mathbb{Z})$ which together with the corresponding torsion subgroups span the entire group, and γ_k, γ'_k are corresponding geometric torsion bases. Then there exists an orbifold diffeomorphism $g: M'/T \rightarrow M/T$ such that $\mu'_h = \mu_h \circ g_*$.*

Proof. By assumption the $\mathcal{G}_{(g; \vec{o})}$ -orbits of the $(2g + m)$ -tuples $((\mu_h(\alpha'_i), \mu_h(\beta'_i))_{i=1}^g, (\mu_h(\gamma'_k))_{k=1}^m)$ and $((\mu'_h(\alpha'_i), \mu'_h(\beta'_i))_{i=1}^g, (\mu'_h(\gamma'_k))_{k=1}^m)$ are equal, and hence $\mu'_h = \mu_h \circ G$, where G is the isomorphism defined by $G(\alpha_i) = \alpha'_i$, $G(\beta_i) = \beta'_i$ and $G(\gamma_i) = \gamma'_{\tau(i)}$. By its definition G is symplectic and torsion geometric, i.e. $G \in \mathrm{Sp}(K', K) \cap S^{K', K}$, where $K := H_1^{\mathrm{orb}}(M/T, \mathbb{Z})$ and $K' := H_1^{\mathrm{orb}}(M'/T, \mathbb{Z})$. Now the result follows from Proposition 4.19. □

4.3 Uniqueness

4.3.1 List of ingredients of (M, σ, T)

We start by assigning a list of invariants to (M, σ) .

Definition 4.27 Let (M, σ) be a compact connected $2n$ -dimensional symplectic manifold equipped with an effective symplectic action of a $(2n - 2)$ -dimensional torus T for which at least one, and hence every T -orbit is a $(2n - 2)$ -dimensional symplectic submanifold of M . The *list of ingredients of (M, σ, T)* consists of the following items.

- 1) The Fuchsian signature $(g; \vec{o}) \in \mathbb{Z}^{1+m}$ of the orbisurface M/T (c.f. Remark 2.14 and Definition 4.20).
- 2) The total symplectic area of the symplectic orbisurface $(M/T, \nu)$, where the symplectic form ν is defined by the condition $\pi^*\nu|_{\Omega_x} = \sigma|_{\Omega_x}$ for every $x \in M$, where $\pi: M \rightarrow M/T$ is the projection map and $\Omega_x = (\mathrm{T}_x(T \cdot x))^{\sigma_x}$ (c.f. Lemma 2.28).
- 3) The non-degenerate antisymmetric bilinear form $\sigma^t: \mathfrak{t} \times \mathfrak{t} \rightarrow \mathbb{R}$ on the Lie algebra \mathfrak{t} of T such that for all $X, Y \in \mathfrak{t}$ and all $x \in M$ $\sigma_x(X_M(x), Y_M(x)) = \sigma^t(X, Y)$ (c.f. Lemma 2.1).
- 4) The monodromy invariant of (M, σ, T) , i.e. the $\mathcal{G}_{(g; \vec{o})}$ -orbit

$$\mathcal{G}_{(g, \vec{o})} \cdot ((\mu_h(\alpha_i), \mu_h(\beta_i))_{i=1}^g, (\mu_h(\gamma_k))_{k=1}^m),$$

of the $(2g + m)$ -tuple $((\mu_h(\alpha_i), \mu_h(\beta_i))_{i=1}^g, (\mu_h(\gamma_k))_{k=1}^m)$, c.f. Definition 4.24.

∅

Theorem 4.28. Suppose that G is the first integral orbifold homology group of a compact, connected, orientable smooth orbisurface of Fuchsian signature $(g; \vec{o}) \in \mathbb{Z}^{1+m}$. Choose a set of generators $\{\alpha_i, \beta_i\}_{i=1}^g$ of a maximal⁷ free subgroup and let $\{\gamma_k\}_{k=1}^m$ be a geometric torsion basis. The group of geometric isomorphisms of G , c.f. Definition 4.9, is equal to the group of isomorphisms of G induced by linear isomorphisms f_B of \mathbb{Z}^{2g+m} where $B \in \mathcal{G}_{(g; \vec{o})}$, and $\mathcal{G}_{(g; \vec{o})}$ is given in Definition 4.22. (Recall that we had to choose the generators $\alpha_i, \beta_i, \gamma_k$ in order to define an endomorphism f_B of \mathbb{Z}^{2g+m} from the matrix B).

Proof. After fixing a group, the statement corresponds to that of Theorem 4.18 formulated in the language of matrices. □

4.3.2 Uniqueness Statement

The next two results say that the list of ingredients of (M, σ, T) as in Definition 4.27 is a complete set of invariants of (M, σ, T) .

Lemma 4.29. Let (M, σ) be a compact connected $2n$ -dimensional symplectic manifold equipped with an effective symplectic action of a $(2n - 2)$ -dimensional torus T for which at least one, and hence every T -orbit is a $(2n - 2)$ -dimensional symplectic submanifold of (M, σ) . Then if (M', σ') is a compact connected $2n$ -dimensional symplectic manifold equipped with an effective symplectic action of T for which at least one, and hence every T -orbit is a $(2n - 2)$ -dimensional symplectic submanifold of (M, σ') , and (M', σ') is T -equivariantly symplectomorphic to (M, σ) , then the list of ingredients of (M', σ', T) is equal to the list of ingredients of (M, σ, T) .

The proof of Lemma 4.29 is analogous to the proof of Lemma 3.11.

⁷This means that together with the torsion subgroup span the entire group.

Proposition 4.30. *Let (M, σ) be a compact connected $2n$ -dimensional symplectic manifold which is equipped with an effective symplectic action of a $(2n - 2)$ -dimensional torus T for which at least one, and hence every T -orbit is a $(2n - 2)$ -dimensional symplectic submanifold. Then if (M', σ') is a compact connected $2n$ -dimensional symplectic manifold equipped with an effective symplectic action of T for which at least one, and hence every T -orbit is a $(2n - 2)$ -dimensional symplectic submanifold of (M', σ') and the list of ingredients of (M', σ', T) is equal to the list of ingredients of (M, σ, T) , then (M', σ') is T -equivariantly symplectomorphic to (M, σ) .*

Proof. Suppose that the list of ingredients of (M, σ) equals the list of ingredients of (M', σ') . Because M/T and M'/T have the same Fuchsian signature and symplectic area, by the orbifold version of Moser's theorem [38, Th. 3.3], the (compact, connected, smooth, orientable) orbisurfaces $(M/T, \nu)$ and $(M'/T, \nu')$ are symplectomorphic, where ν and ν' are the symplectic forms given by Lemma 2.28.

Let μ, μ', μ_h, μ'_h be the monodromy homomorphisms from the orbifold fundamental groups $\pi_1^{\text{orb}}(M/T, p_0), \pi_1^{\text{orb}}(M'/T, p_0)$, and from the first integral orbifold homology groups $H_1^{\text{orb}}(M/T, \mathbb{Z}), H_1^{\text{orb}}(M'/T, \mathbb{Z})$ into the torus T , respectively associated to the symplectic manifolds $(M, \sigma), (M', \sigma')$ as in Definition 2.46. Because ingredient 4) of (M, σ) equals ingredient 4) of (M', σ') , by Proposition 4.26 and Definition 4.9 there exists an orbifold diffeomorphism $F: M/T \rightarrow M'/T$ such that $\mu_h = \mu'_h \circ F_*$, and hence $\mu = \mu' \circ F_*$.

The remaining part of the proof is analogous to the proof in the free case, “proof of Proposition 3.12”, and the details of the arguments that follow may be found there. If $\nu_0 := F^*\nu'$, the symplectic orbit space $(M/T, \nu_0)$ is symplectomorphic to $(M'/T, \nu')$, by means of F . Let $\tilde{\nu}_0$ be the pullback of the 2-form ν_0 by the orbifold universal cover $\psi: \widetilde{M/T} \rightarrow M/T$ of M/T based at $p_0 = \psi(x_0)$, and similarly we define $\tilde{\nu}'$ by means of the universal cover $\psi': \widetilde{M'/T} \rightarrow M'/T$ based at $F(p_0)$. As in the proof of Proposition 3.12, the orbifold symplectomorphism F between $(M/T, \nu_0)$ and $(M'/T, \nu')$ lifts to a unique symplectomorphism \tilde{F} between $(\widetilde{M/T}, \tilde{\nu}_0)$ and $(\widetilde{M'/T}, \tilde{\nu}')$ such that $\tilde{F}(x_0) = x'_0$. Then the assignment

$$[[\gamma], t]_{\pi_1^{\text{orb}}(M/T, p_0)} \mapsto [\tilde{F}([\gamma]), t]_{\pi_1^{\text{orb}}(M'/T, p'_0)},$$

is a T -equivariant symplectomorphism between the associated bundle $\widetilde{M/T} \times_{\pi_1^{\text{orb}}(M/T, p_0)} T$ and the associated bundle $\widetilde{M'/T} \times_{\pi_1^{\text{orb}}(M'/T, p'_0)} T$, which by Theorem 2.36, gives a T -equivariant symplectomorphism between (M, σ) and (M', σ') . \square

4.4 Existence

4.4.1 List of ingredients for T

We assign to a torus T a list of four ingredients. This is analogous to Definition 3.13.

Definition 4.31 Let T be a torus. The *list of ingredients for T* consists of the following.

- i) An $(m + 1)$ -tuple $(g; \vec{o})$ of integers, where \vec{o} is non-decreasingly ordered and consists of strictly positive integers and m is a non-negative integer, and such that $(g; \vec{o})$ is not of the form $(0; o_1)$ or of the form $(0; o_1, o_2)$ with $o_1 < o_2$.

- ii) A positive real number $\lambda > 0$.
- iii) A non-degenerate antisymmetric bilinear form $\sigma^t: \mathfrak{t} \times \mathfrak{t} \rightarrow \mathbb{R}$ on the Lie algebra \mathfrak{t} of T .
- iv) An orbit $\mathcal{G}_{(g, \vec{o})} \cdot \xi \in T_{(g, \vec{o})}^{2g+m} / \mathcal{G}_{(g, \vec{o})}$ in the Fuchsian signature space associated to $(g; \vec{o})$, where $T_{(g, \vec{o})}^{2g+m} = \{(t_i)_{i=1}^{2g+m} \in T^{2g+m} \mid \text{the order of } t_i \text{ is a multiple of } o_i, 2g+1 \leq i \leq 2g+m\}$, and where $\mathcal{G}_{(g, \vec{o})}$ is the group of matrices in Definition 4.22.

∅

4.4.2 Existence statement

Any list of ingredients as in Definition 4.31 gives rise to one of our manifolds with symplectic T -action.

Proposition 4.32. *Let T be a $(2n-2)$ -dimensional torus. Then given a list of ingredients for T , as in Definition 4.31, there exists a $2n$ -dimensional symplectic manifold (M, σ) with an effective symplectic action of T for which at least one, and hence every T -orbit is a $(2n-2)$ -dimensional symplectic submanifold of (M, σ) , and such that the list of ingredients of (M, σ, T) is equal to the list of ingredients for T .*

Proof. Let \mathcal{I} be a list of ingredients for the torus T , as in Definition 4.31. Let the pair (Σ, σ^Σ) be a compact, connected symplectic orbisurface of Fuchsian signature $(g; \vec{o})$ given by ingredient i) of \mathcal{I} in Definition 4.31, and with total symplectic area equal to the positive real number λ , where λ is given by ingredient ii) of \mathcal{I} . By [55, Th. 13.3.6], since $(g; \vec{o})$ is not of the form $(0; o_1), (0; o_1, o_2)$ with $o_1 < o_2$ by assumption, Σ is a (very) good orbisurface. Let the space $\tilde{\Sigma}$ be the orbifold universal cover of Σ , which is a smooth manifold, based at an arbitrary regular point $p_0 \in \Sigma$ which we fix for the rest of the proof, c.f. the construction we gave prior to Definition 2.32 and Theorem 2.36. Because Σ is very good orbisurface, $\tilde{\Sigma}$ is a smooth surface. Let α_i, β_i , for $1 \leq i \leq 2g$, be a symplectic basis of a free subgroup of $H_1^{\text{orb}}(\Sigma, \mathbb{Z})$ which together with the torsion subgroup span the entire group, and let γ_k , for $1 \leq k \leq m$, be a geometric torsion basis and define $\mu_h: H_1^{\text{orb}}(\Sigma, \mathbb{Z}) \rightarrow T$ to be the unique homomorphism such that $\mu_h(\alpha_i) = a_i, \mu_h(\beta_i) = b_i, \mu_h(\gamma_k) = c_k$, where the tuple $((a_i, b_i), (\gamma_k))$ represents ingredient iv) of \mathcal{I} , and h_1 denotes the orbifold Hurewicz homomorphism from $\pi_1^{\text{orb}}(\Sigma, p_0)$ to $H_1^{\text{orb}}(\Sigma, \mathbb{Z})$. Let $\mu: \pi_1^{\text{orb}}(\Sigma, p_0) \rightarrow T$ be the homomorphism defined as $\mu := \mu_h \circ h_1$. Let the orbifold fundamental group $\pi_1^{\text{orb}}(\Sigma, p_0)$ act on the smooth manifold given as the Cartesian product $\tilde{\Sigma} \times T$ by the diagonal action $[\delta] \cdot ([\gamma], t) = ([\delta \gamma^{-1}], \mu([\delta]) t)$. Define the bundle space

$$M_{\text{model}, p_0}^\Sigma := \tilde{\Sigma} \times_{\pi_1^{\text{orb}}(\Sigma, p_0)} T. \quad (4.26)$$

The symplectic form and torus action on $M_{\text{model}, p_0}^\Sigma$ are constructed in the exact same way as in the free case, c.f. proof of Proposition 3.14.

The proof that ingredients 1)–3) of $(M_{\text{model}, p_0}^\Sigma, \sigma_{\text{model}}^\Sigma)$ are equal to ingredients 1)–3) of the list \mathcal{I} is the same as in the free case (c.f. proof of Proposition 3.14), with the observation that

in the non-free case we use the classification theorem of compact, connected, smooth orientable orbisurfaces Theorem 4.2 of Thurston's instead of the classical classification theorem of compact, connected, smooth surfaces, to prove that the corresponding ingredients 1) agree. We have left to show that ingredient 4) of $(M_{\text{model}}^\Sigma, p_0, \sigma_{\text{model}}^\Sigma)$ equals ingredient 4) of the list \mathcal{I} . Let $\Omega_{\text{model}}^\Sigma$ stand for the flat connection on M_{model}^Σ given by the symplectic orthogonal complements to the tangent spaces to the T -orbits, see Proposition 2.17, and let $\mu_h^{\Omega_{\text{model}}^\Sigma}$ stand for the induced homomorphism $\mu_h^{\Omega_{\text{model}}^\Sigma}: H_1^{\text{orb}}(M_{\text{model}}^\Sigma/T, \mathbb{Z}) \rightarrow T$ in homology, by the monodromy of such connection. If $f: H_1^{\text{orb}}(M_{\text{model}}^\Sigma/T, \mathbb{Z}) \rightarrow H_1^{\text{orb}}(\Sigma, \mathbb{Z})$ is the group isomorphism induced by the orbifold symplectomorphism

$$M_{\text{model}}^\Sigma/T \rightarrow \tilde{\Sigma}/\pi_1^{\text{orb}}(\Sigma, p_0) \rightarrow \Sigma, \quad (4.27)$$

where each arrow in (4.27) represents the natural map,

$$\mu_h^{\Omega_{\text{model}}^\Sigma} = \mu_h \circ f. \quad (4.28)$$

Because f is induced by a diffeomorphism, by Theorem 4.18, f is symplectic and torsion geometric, c.f. Definition 4.14 and Definition 4.13. Therefore there exists a unique collection of elements $\alpha'_i, \beta'_i, 1 \leq i \leq g$, in $H_1^{\text{orb}}(M_{\text{model}}^\Sigma/T, \mathbb{Z})$ such that $f(\alpha'_i) = \alpha_i$ and $f(\beta'_i) = \beta_i$, for all $1 \leq i \leq g$. The elements $\alpha'_i, \beta'_i, 1 \leq i \leq g$, form a symplectic basis of a free subgroup $F^{\Omega_{\text{model}}^\Sigma}$ of the orbifold homology group $H_1^{\text{orb}}(M_{\text{model}}^\Sigma/T, \mathbb{Z})$, which together with the torsion subgroup spans the entire group. Similarly let the collection γ'_k , for $1 \leq k \leq m$, be such that $f(\gamma'_k) = \gamma_k$, for all $1 \leq k \leq m$. The $\gamma'_k, 1 \leq k \leq m$, form a geometric torsion basis, c.f. Definition 4.4 such that $\omega_k = \omega'_{\tau(k)}$ for all $k, 1 \leq k \leq m$, for a permutation $\tau \in S_m^{\vec{o}}$. Let $\hat{\xi}$ be the $(2g + m)$ -tuple of elements $\mu_h^{\Omega_{\text{model}}^\Sigma}(\alpha'_i), \mu_h^{\Omega_{\text{model}}^\Sigma}(\beta'_i), \mu_h^{\Omega_{\text{model}}^\Sigma}(\gamma'_k)$, where $1 \leq i \leq g$ and $1 \leq k \leq m$. Therefore by (4.28)

$$\hat{\xi} = ((\mu_h(\alpha_i), \mu_h(\beta_i))_{i=1}^g, (\mu_h(\gamma_k))_{k=1}^m),$$

which in particular implies that ingredient 4) of $(M_{\text{model}}, p_0, \sigma_{\text{model}}^\Sigma)$ equals $((a_i, b_i), c_k)$. \square

4.5 Classification Theorem

By putting together the results of the previous subsections, we obtain the main result of the section:

Theorem 4.33. *Let T be a $(2n - 2)$ -dimensional torus. Let (M, σ) be a compact connected $2n$ -dimensional symplectic manifold on which T acts effectively and symplectically and such that at least one, and hence every T -orbit is a $(2n - 2)$ -dimensional symplectic submanifold of (M, σ) .*

Then the list of ingredients of (M, σ, T) as in Definition 4.27 is a complete set of invariants of (M, σ, T) , in the sense that, if (M', σ') is a compact connected $2n$ -dimensional symplectic manifold equipped with an effective symplectic action of T for which at least one, and hence every T -orbit is a $(2n - 2)$ -dimensional symplectic submanifold of (M', σ') , (M', σ') is T -equivariantly symplectomorphic to (M, σ) if and only if the list of ingredients of (M', σ', T) is equal to the list of ingredients of (M, σ, T) .

And given a list of ingredients for T , as in Definition 4.31, there exists a symplectic $2n$ -dimensional manifold (M, σ) with an effective symplectic torus action of T for which at least one, and hence every T -orbit is a $(2n - 2)$ -dimensional symplectic submanifold of (M, σ) , such that the list of ingredients of (M, σ, T) is equal to the list of ingredients for T .

Proof. It follows by putting together Lemma 4.29, Proposition 4.30 and Proposition 4.32. The combination of Lemma 4.29, Proposition 4.30 gives the existence part of the theorem, while Proposition 4.32 gives the existence part. \square

Remark 4.34 The author is grateful to professor Deligne for pointing out an imprecision in a earlier version of the following statements. Let T be a $(2n - 2)$ -dimensional torus. Let \mathcal{M} denote the category of which the objects are the compact connected symplectic $2n$ -dimensional manifolds (M, σ) together with an effective symplectic T -action on (M, σ) such that at least one, and hence every T -orbit is a $(2n - 2)$ -dimensional symplectic submanifold of M , and of which the morphisms are the T -equivariant symplectomorphisms of (M, σ) . Let \mathcal{I} denote the set of all lists of ingredients as in Definition 4.31, viewed as a category, and of which the identities are the only endomorphisms of categories. Then the assignment ι in Definition 4.27 is a full functor categories from the category \mathcal{M} onto the category \mathcal{I} . In particular the proper class \mathcal{M}/\sim of isomorphism classes in \mathcal{M} is a set, and the functor $\iota: \mathcal{M} \rightarrow \mathcal{I}$ in Definition 4.27 induces a bijective mapping ι/\sim from \mathcal{M}/\sim onto \mathcal{I} . The fact that the mapping $\iota: \mathcal{M} \rightarrow \mathcal{I}$ is a functor and the mapping ι/\sim is injective follows from the uniqueness part of the statement of Theorem 4.33. The surjectivity of ι , follows from the existence part of the statement of Theorem 4.33. \oslash

5 The four dimensional classification

We give a classification of effective symplectic actions of 2-tori on compact, connected, symplectic 4-dimensional manifolds under no additional assumption.

This generalizes the 4-dimensional case of Delzant's theorem on the classification of symplectic-toric manifolds to symplectic actions which are not Hamiltonian.

5.1 Classification Statement

We state the main theorem.

The following is a special case of [12, Def. 9.1]. We call the list “free and Lagrangian” for obvious reasons, c.f. Theorem 5.6.

Definition 5.1 Let T be a 2-torus. A *free and Lagrangian list of ingredients for T* consists of:

- 1) A discrete cocompact subgroup P of \mathfrak{t}^* .
- 2) An antisymmetric bilinear mapping $c: \mathfrak{t}^* \times \mathfrak{t}^* \rightarrow \mathfrak{t}$ with the following properties.
 - 2a) For every $\zeta, \zeta' \in P$, the element $c(\zeta, \zeta') \in \mathfrak{t}$ belongs to the integral lattice $T_{\mathbb{Z}}$ in \mathfrak{t} , the kernel of the exponential mapping $\exp: \mathfrak{t} \rightarrow T$.

2b) For every $\zeta, \zeta', \zeta'' \in \mathfrak{t}^*$ we have that

$$\zeta(c(\zeta', \zeta'')) + \zeta'(c(\zeta'', \zeta)) + \zeta''(c(\zeta, \zeta')) = 0.$$

3) An element in \mathcal{T} , c.f. (5.3).

○

The following definition is a particular case of [12, Def. 7.9].

Definition 5.2 Let $c: \mathfrak{t}^* \times \mathfrak{t}^* \rightarrow \mathfrak{t}$ be as in item 2) of Definition 5.1. Let $\text{Hom}_c(P, T)$ denote the space of mappings $\tau: P \rightarrow T, \zeta \mapsto \tau_\zeta$, such that

$$\tau_{\zeta'} \tau_\zeta = \tau_{\zeta+\zeta'} e^{c(\zeta', \zeta)/2}, \quad \zeta, \zeta' \in P. \quad (5.1)$$

For each $\zeta' \in \mathfrak{t}^*, \zeta \mapsto c(\zeta, \zeta')$ is a homomorphism from P to \mathfrak{t} . Write $c(\cdot, \mathfrak{t}^*)$ for the set of all $c(\cdot, \zeta') \in \text{Hom}(P, \mathfrak{t})$ such that $\zeta' \in \mathfrak{t}^*$. $c(\cdot, \mathfrak{t}^*)$ is a linear subspace of the Lie algebra $\text{Hom}(P, \mathfrak{t})$ of $\text{Hom}(P, T)$. If $h: \zeta \mapsto h_\zeta$ is a homomorphism from P to T , then $h \cdot \tau: \zeta \mapsto \tau_\zeta h_\zeta \in \text{Hom}_c(P, T)$ for every $\tau \in \text{Hom}_c(P, T)$, and $(h, \tau) \mapsto h \cdot \tau$ defines a free, proper, and transitive action of $\text{Hom}(P, T)$ on $\text{Hom}_c(P, T)$. Let Sym denote the space of all linear mappings $\alpha: \mathfrak{t}^* \rightarrow \mathfrak{t}, \xi \mapsto \alpha_\xi$, which are symmetric in the sense of

$$\xi(\alpha_{\xi'}) - \xi'(\alpha_\xi) = 0. \quad (5.2)$$

For each $\alpha \in \text{Sym}$, the restriction $\alpha|_P$ of α to P is a homomorphism from P to \mathfrak{t} . In this way the set $\text{Sym}|_P$ of all $\alpha|_P$ such that $\alpha \in \text{Sym}$ is another linear subspace of $\text{Hom}(P, \mathfrak{t})$. Write

$$\mathcal{T} := \text{Hom}_c(P, T) / \exp \mathcal{A}, \quad \mathcal{A} := c(\cdot, \mathfrak{t}^*) + \text{Sym}|_P \quad (5.3)$$

for the orbit space of the action of the Lie subgroup $\exp \mathcal{A}$ of $\text{Hom}(P, T)$ on $\text{Hom}_c(P, T)$. ○

Remark 5.3 Because $c(P \times P) \subset T_{\mathbb{Z}}$, the factor $e^{c(\zeta', \zeta)/2}$ in (5.1) is an element of order two in T . The elements of $\text{Hom}_c(P, T)$ are homomorphisms from P to T if $c(P \times P) \subset 2T_{\mathbb{Z}}$. $\text{Hom}(P, T)$ is a torus group, a compact, connected, and commutative Lie group with Lie algebra equal to the vector space $\text{Hom}(P, \mathfrak{t})$ of dimension $(\dim T)^2$. $\text{Hom}_c(P, T)$ is diffeomorphic to a torus of dimension $(\dim T)^2$. ○

The following is our main result of this paper, a classification of symplectic actions of 2-tori on compact, connected symplectic 4-manifolds, up to T -equivariant symplectomorphisms. Theorem 5.4 consists of two separate statements. The first statement says that the examples in i) and ii) are symplectic T -manifolds with certain characteristics. Moreover it says that any symplectic 4-manifold with a 2-torus action is of the form given in i), ii), 1) or 2). The second statement says that for any two different lists of ingredients (a discrete cocompact subgroup, a bilinear form etc.) as in i) or ii) give rise to symplectic manifolds which are not T -equivariantly symplectomorphic. Hence Theorem 5.4 is a full classification.

Theorem 5.4. *Let T be a 2-dimensional torus. Then the following hold:*

- i) *For any choice of a discrete cocompact subgroup P of \mathfrak{t}^* , a bilinear form $c: \mathfrak{t}^* \times \mathfrak{t}^* \rightarrow \mathfrak{t}$ and an equivalence class of a mapping $[\tau]_{\exp(\mathcal{A})}$, as in Definition 5.1, let $\iota: P \rightarrow T \times \mathfrak{t}^*$ be the mapping $\zeta \mapsto (\tau_\zeta^{-1}, \zeta)$, which is a homomorphism onto a discrete cocompact subgroup of $T \times \mathfrak{t}^*$ with respect to the non-abelian group structure given by*

$$(t, \zeta)(t', \zeta') = (t t' e^{-c(\zeta, \zeta')/2}, \zeta + \zeta').$$

Equip $T \times \mathfrak{t}^$ with the standard cotangent bundle symplectic form. Then $(T \times \mathfrak{t}^*)/\iota(P)$ equipped with the action of T which comes from the action of T by translations on the left factor of $T \times \mathfrak{t}^*$, and where the symplectic form on $(T \times \mathfrak{t}^*)/\iota(P)$ is the T -invariant form induced by the symplectic form on $T \times \mathfrak{t}^*$, is a compact, connected symplectic 4-manifold on which T acts freely and for which the T -orbits are Lagrangian 2-tori.*

- ii) *For any choice of an $(m+1)$ -tuple $(g; \vec{o})$ of integers, a positive real number $\lambda > 0$, a non-degenerate antisymmetric bilinear form σ^\dagger on \mathfrak{t} , and an element $\mathcal{G}_{(g, \vec{o})} \cdot \xi \in T_{(g, \vec{o})}^{2g+m}/\mathcal{G}_{(g, \vec{o})}$ where $\xi = ((a_i, b_i), (c_k))$ as in Definition 4.31, let Σ be an orbisurface with Fuchsian signature $(g; \vec{o})$, and total symplectic area λ , and let $p_0 \in \Sigma$. Let α_i, β_i , for $1 \leq i \leq 2g$, be a symplectic basis of a free subgroup of $H_1^{\text{orb}}(\Sigma, \mathbb{Z})$ which together with the torsion subgroup spans $H_1^{\text{orb}}(\Sigma, \mathbb{Z})$, and let γ_k , for $1 \leq k \leq m$, be a geometric torsion basis (c.f. Definition 4.4). Let h_1 be the Hurewicz homomorphism. Let f_h to be the unique homomorphism such that $f_h(\alpha_i) = a_i$, $f_h(\beta_i) = b_i$, $f_h(\gamma_k) = c_k$. Let $f := f_h \circ h_1$. Let $\pi_1^{\text{orb}}(\Sigma, p_0)$ act on $\tilde{\Sigma} \times T$ by*

$$[\delta] \cdot ([\gamma], t) = ([\delta \gamma^{-1}], f([\delta]) t).$$

Equip the universal cover $\tilde{\Sigma}$ with the symplectic form pullback from Σ , and $\tilde{\Sigma} \times T$ with the product symplectic form. Let T act by translations on the right factor of $\tilde{\Sigma} \times T$. Then the space $\tilde{\Sigma} \times_{\pi_1^{\text{orb}}(\Sigma, p_0)} T$ endowed with the unique symplectic form and T -action induced by the product ones is a compact, connected symplectic 4-manifold on which T acts effectively and for which the T -orbits are symplectic 2-tori.

Moreover, if (M, σ) is a compact connected symplectic 4-dimensional manifold equipped with an effective symplectic action of a 2-torus T , then one and only one of the following cases occurs:

- 1) *(M, σ) is a 4-dimensional symplectic toric manifold, hence determined up to T -equivariant symplectomorphisms by its Delzant polygon $\mu(M)$ centered at the origin, where $\mu: M \rightarrow \mathfrak{t}^*$ is the momentum map for the T -action.*
- 2) *(M, σ) is equivariantly symplectomorphic to a product $\mathbb{T}^2 \times S^2$, where $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$ and the first factor of \mathbb{T}^2 acts on the left factor by translations on one component, and the second factor acts on S^2 by rotations about the vertical axis of S^2 . The symplectic form is a positive linear combination of the standard translation invariant form on \mathbb{T}^2 and the standard rotation invariant form on S^2 .*
- 3) *(M, σ) is T -equivariantly symplectomorphic to one and only one of the symplectic T -manifolds in part i), given by a unique list of ingredients.*

- 4) (M, σ) is T -equivariantly symplectomorphic to one and only one of the symplectic T -manifolds in part ii), given by a unique list of ingredients.

5.2 Proof of Theorem 5.4

We make use of results in sections 3, 4 and ideas/methods in proofs of [12, Prop. 5.5, Lem. 7.1, Lem. 7.5]. Because we do not reprove anything which appeared therein, it is unrealistic to expect understanding every detail of the proof below without prior knowledge of them. References are given to specific parts of the previous sections and to [12] when appropriate. We divide the proof into three steps.

Step 1. First suppose that the 2-dimensional T -orbits which are Lagrangian submanifolds of (M, σ) . The ingredients of the construction of the model $(M_{\text{model}}, \sigma_{\text{model}})$ in Claim 1.1 below, which we obtain as a particular case of [12, Prop. 7.2, Prop. 7.4], are the following. (M_h, σ_h) is a Delzant submanifold of (M, σ) , where σ_h equals σ restricted to M_h ; T_h is the maximal subtori of T which acts on M in a Hamiltonian fashion. Because T is 2-dimensional, it follows from [12, Lem. 2.6, Lem. 3.11] that $T = T_h$ if the T -action is Hamiltonian, $T = \{1\}$ if the action is free, and otherwise $T_x = T_h$ for any x such that $T_x \neq \{1\}$; $N = (\mathfrak{t}/\mathfrak{t}_h)^*$; G is the Lie group

$$G := T \times N, \quad (5.4)$$

which is two-step nilpotent with respect to the non-abelian product

$$(t, \zeta)(t', \zeta') = (t t' e^{-c(\zeta, \zeta')/2}, \zeta + \zeta'), \quad (5.5)$$

where

$$c: N \times N \rightarrow \mathfrak{t}, \quad (5.6)$$

is a certain antisymmetric bilinear form which represents the Chern class of the principal bundle $M_{\text{reg}} \rightarrow M_{\text{reg}}/T$; the so called period lattice

$$P \leq N, \quad (5.7)$$

which is a discrete, cocompact, subgroup; $H \leq G$ is a commutative, closed, Lie subgroup, defined in terms of the holonomy τ , which lives in certain exponential quotient of the, roughly speaking, homomorphism group $\text{Hom}_c(P, T)$, c.f. (5.3); H acts on $G \times M_h$ by

$$((t, \zeta), x) \mapsto (t \tau_\zeta) \cdot x. \quad (5.8)$$

In a formula,

$$H = \{(t, \zeta) \mid \xi \in P, t \tau_\zeta \in T_h\} \leq G. \quad (5.9)$$

The map

$$((t, \zeta), x) \mapsto (t\tau_\zeta) \cdot x : H \times M_h \rightarrow M_h \quad (5.10)$$

defines a smooth action of H on the Delzant manifold M_h . The symplectic form σ_{model} on $M_{\text{model}} := G \times_H M_h$ is given by an explicit formula as follows, c.f. [12, Prop. 7.4], where $X \mapsto X_h$ is a projection onto \mathfrak{t}_h (this depends on a choice of freely acting torus T_f , which induces a decomposition of \mathfrak{t} , c.f. [12, Sec. 5.2]). First consider the form on $G \times M_h$

$$\begin{aligned} \omega_a(\delta a, \delta' a) &= \delta\zeta(X') - \delta'\zeta(X) \\ &\quad - \mu(x)(c_h(\delta\zeta, \delta'\zeta)) + (\sigma_h)_x(\delta x, (X'_h)_{M_h}(x)) \\ &\quad - (\sigma_h)_x(\delta' x, (X_h)_{M_h}(x)) + (\sigma_h)_x(\delta x, \delta' x), \end{aligned} \quad (5.11)$$

where c_h is the \mathfrak{t}_h -component of c and μ is the momentum map for the Hamiltonian T_h -action, c.f. [12, Lem. 3.11]. Here $\delta a = ((\delta t, \delta\zeta), \delta x)$ and $\delta' a = ((\delta't, \delta'\zeta), \delta' x)$ are tangent vectors to $G \times M_h$ at $a = ((t, \zeta), x)$, where we identify each tangent space of the torus T with \mathfrak{t} , and $X = \delta t + c(\delta\zeta, \zeta)/2$ and $X' = \delta't + c(\delta'\zeta, \zeta)/2$. Let $\pi_{M_{\text{model}}} : G \times M_h \rightarrow M_{\text{model}}$ be the canonical projection. Then there exists a unique 2-form σ_{model} such that

$$\omega = \pi_{M_{\text{model}}}^* \sigma_{\text{model}}. \quad (5.12)$$

The translational T -action on M_{model} by left multiplications on T is so that T_h acts on the Delzant manifold M_h , and its complement T_f in T , which acts freely, permutes the Delzant submanifolds.

Claim 1.1. There exists a T -equivariant symplectomorphism from $(M_{\text{model}}, \sigma_{\text{model}})$ onto (M, σ) .

Proof of Claim 1.1.(Sketch). The first observation is that the orbit space M/T is a polyhedral \mathfrak{t}^* -parallel space. A \mathfrak{t}^* -parallel space is a Hausdorff, topological space modelled on a corner of \mathfrak{t}^* , c.f. [12, Def. 10.1]. The local charts ϕ_α into \mathfrak{t}^* , satisfy that the mapping $x \mapsto \phi_\alpha(x) - \phi_\beta(x)$ has to be locally constant for all values of α and β . Consider the form $\hat{\sigma}(X) := -i_{X_M} \sigma \in \Omega^1(M)$, which is a closed, basic form, if $X \in \mathfrak{t}$. The assignment $\hat{\sigma} : x \mapsto (\hat{\sigma}_x : T_x M \rightarrow \mathfrak{t}^*)$, where we use the identification $\hat{\sigma}_x \simeq T_x \pi$, induces an isomorphism $\hat{\sigma}_p : T_p(M_{\text{reg}}/T) \rightarrow \mathfrak{t}^*$. This implies that a “constant vector field on $\mathcal{X}^\infty(M_{\text{reg}}/T)$ ” can be thought of as an element “ $\xi \in \mathfrak{t}^{**}$ ”. L_ξ is a lift of ξ if $\hat{\sigma}_x(L_\xi) = \xi$. Secondly, as \mathfrak{t}^* -parallel spaces, there is an isomorphism $M/T \simeq \Delta \times S$, where Δ is a Delzant polytope, and S is a torus, c.f. [12, Prop. 3.8, Th. 10.12]. Proving this involves the classification of V -parallel spaces. Moreover, it involves generalizing the Tietze–Nakajima theorem in [52], [39]. Here the Delzant polytope Δ captures the Hamiltonian part of the action, while the torus S captures the free part.

Recall that M_{reg} is the subset of M where the T -action is free. In [12, Prop. 5.5] we showed that there exists a “nice” admissible connection $\xi \in \mathfrak{t}^* \mapsto L_\xi \in \mathcal{X}^\infty(M_{\text{reg}})$, [12, Def. 5.3], for the principal T -bundle $\pi : M_{\text{reg}} \rightarrow M_{\text{reg}}/T$. The lifts L_ξ , $\xi \in N$, have smooth extensions to M . Here “nice” means with simple Lie brackets $[L_\xi, L_\eta]$, such as $[L_\xi, L_\eta] = c(\xi, \eta)_M$, $\xi, \eta \in N$, where c as in (5.6) represents the Chern class of $\pi : M_{\text{reg}} \rightarrow M_{\text{reg}}/T$, and zero otherwise. We also achieve simple symplectic pairings $\sigma(L_\xi, L_\eta)$. In the particular case that $c_h = 0$, we have that $\sigma(L_\xi, L_\eta) = 0$, for all $\xi, \eta \in \mathfrak{t}^*$. The lifts L_ξ , $\xi \notin N$, are singular on $M \setminus M_{\text{reg}}$. The singularities are also required to be simple. This step involves computations in the cohomology ring $H^*(M/T)$.

Then in [12, Prop. 6.1] we define the integrable distribution $\{D_x\}_{x \in M}$ on M where D_x is the span of $L_\eta(x)$, $Y_M(x)$ as $Y \in \mathfrak{t}_h$, $\eta \in C$, where $C \oplus N = \mathfrak{t}^*$. The integral manifolds of this distribution are T -equivariantly symplectomorphic to the Delzant manifold (M_h, σ_h, T_h) . Assuming this, we make the definition of G in (5.4), H in (5.9), and M_{model} from the connection $\xi \mapsto L_\xi$, and the definition of σ_{model} in (5.12). The definition of H involves the holonomy of the connection $\xi \mapsto L_\xi$. And the T -equivariant symplectomorphism between the model $G \times_H M_h$ and our symplectic T -manifold (M, σ) is induced by

$$((t, \xi), x) \mapsto t \cdot e^{L_\xi}(x): G \times M_h \rightarrow M. \quad (5.13)$$

P is rigorously introduced in Lemma 10.12, Proposition 3.8 in [12]; c is introduced in in Proposition 5.5 therein.

Step 2. In Step 1 we introduced ingredients i) and ii) below. The definition of ingredient iii) takes Section 7.5 in [12].⁸

Definition 5.5 Let (M, σ) be a compact, connected symplectic manifold equipped with a free symplectic action of a 2-torus T for which the T -orbits are Lagrangian submanifolds of (M, σ) . The *list of ingredients* of (M, σ, T) , as in Definition 5.1, consists of:

- i) The period lattice group P in \mathfrak{t}^* , where \mathfrak{t}^* acts on the orbit space M/T , c.f. (5.7).
- ii) The antisymmetric bilinear mapping $c: \mathfrak{t}^* \times \mathfrak{t}^* \rightarrow \mathfrak{t}$, c.f. (5.6) with $\mathfrak{t}_h = \{0\}$ and so $N = \mathfrak{t}^*$.
- iii) The so called *holonomy invariant* of (M, σ, T) , which is an element in \mathcal{T} , c.f. (5.3).

□

[12, Th. 9.4, Th. 9.6] imply the following.

Theorem 5.6. *Let T be a 2-torus. The list of ingredients of (M, σ, T) is a complete set of invariants for the compact connected symplectic manifold (M, σ) with free symplectic T -action with Lagrangian orbits, in the following sense.*

If (M', σ') is another compact connected symplectic manifold with a free symplectic T -action with Lagrangian orbits, then there exists a T -equivariant symplectomorphism from (M, σ) onto (M', σ') if and only if the list of ingredients of (M, σ, T) is equal to the list of ingredients of (M', σ', T) .

Moreover, every list of ingredients as in Definition 5.1 is equal to the list of invariants of a compact connected symplectic manifold (M, σ) with free symplectic T -action with Lagrangian orbits.

The next step combines Step 1 and Step 2, together with the results in sections 3, 4.

⁸Recall: the proof of Theorem 5.4 depends on Definition 5.5 below, and in particular in item iii), but its statement did not.

Step 3. Throughout we refer to the items in the statement of Theorem 5.4. It follows from the definitions that the mapping $\iota : \zeta \mapsto (\tau_\zeta^{-1}, \zeta)$ in part i) is a homomorphism from P onto a discrete cocompact subgroup of $T \times \mathfrak{t}^*$. Proving that the space defined in part i) is a compact, connected symplectic 4-manifold equipped with an effective symplectic action is also an exercise using the definitions. Similarly, it follows from the pointwise expression for the symplectic form on $T \times \mathfrak{t}^*$ that the symplectic form on $(T \times \mathfrak{t}^*)/\iota(P)$ vanishes along the T -orbits, which hence are isotropic submanifolds of $(T \times \mathfrak{t}^*)/\iota(P)$. The action on $T \times \mathfrak{t}^*$ is free, and passes to a free action on $(T \times \mathfrak{t}^*)/\iota(P)$, and hence all the T -orbits $(T \times \{t\})/\iota(P)$ are 2-dimensional Lagrangian submanifolds of $(T \times \mathfrak{t}^*)/\iota(P)$, diffeomorphic to T itself. The fact that the space in case ii) is a symplectic manifold equipped with an effective action, and that all the elements involved in the definition are well defined was checked in the proof of Proposition 4.32 and the references therein given. The fact that the T -orbits are 2-tori follows by the same reasoning as in i).

To prove the second part, suppose that (M, σ) is a compact connected symplectic 4-dimensional manifold equipped with an effective symplectic action of a 2-torus T . If there exists a 2-dimensional symplectic T -orbit then, by Lemma 2.4, every T -orbit is a 2-dimensional symplectic submanifold of (M, σ) . Assume that none of the T -orbits is a symplectic 2-dimensional submanifold of (M, σ) . Then the antisymmetric bilinear form σ^t in (2.1) is degenerate, and hence it has a one or two-dimensional kernel $\mathfrak{l} \subset \mathfrak{t}$. If $\dim \mathfrak{l} = 2$, then $\mathfrak{l} = \mathfrak{t}$ and $\sigma^t = 0$, so every T -orbit is an isotropic submanifold of (M, σ) . Hence the 2-dimensional T -orbits are Lagrangian submanifolds of (M, σ) . If $\dim \mathfrak{l} = 1$ there exists a one-dimensional complement V to \mathfrak{l} in \mathfrak{t} , such that the restriction of σ^t to V is a non-degenerate, antisymmetric bilinear form, and hence identically equal to zero, a contradiction. Hence either every T -orbit is a 2-dimensional symplectic submanifold of (M, σ) , or the 2-dimensional T -orbits are Lagrangian submanifolds of (M, σ) . We distinguish three cases according to this.

Case 3.1. Suppose that the action of T on M is Hamiltonian. Because T is 2-dimensional, (M, σ) is a symplectic toric manifold, and hence by Delzant's theorem [10], the image $\mu(M)$ of M under the momentum map $\mu: M \rightarrow \mathfrak{t}^*$ determines (M, σ) up to T -equivariant symplectomorphisms (the explicit construction of M from $\mu(M)$ is given in Delzant's article).

In the next three cases we assume that the T -action on M is not Hamiltonian, so that (M, σ) is not a symplectic toric manifold.

Case 3.2. Suppose that (M, σ) has Lagrangian 2-dimensional orbits, and that T does not act freely on (M, σ) with all T -orbits being Lagrangian 2-tori. By [12, Lem. 2.6] the stabilizer subgroups are connected, and hence subtori of T . Since the T action is not free, there exists a 1-dimensional stabilizer subgroup, and hence the Hamiltonian torus T_h is a 1-dimensional subtori of T . Let T_f be a complementary torus to the Hamiltonian torus T_h in T , and let \mathfrak{t}_f be its Lie algebra. Since \mathfrak{t} is 2-dimensional and $N = (\mathfrak{t}/\mathfrak{t}_h)^*$, $\dim N = 1$. Therefore the antisymmetric bilinear form $c: N \times N \rightarrow \mathfrak{t}$ in (5.6) is identically zero. Then the Lie group G in (5.4) is the Cartesian product $T_h \times G_f$, in which $G_f := T_f \times N$, where the product in G_f is defined by $(t_f, \zeta)(t'_f, \zeta') = (t_f t'_f, \zeta + \zeta')$,

and by [12, Lem. 7.1, iii) form. (7.8)] we have that $\tau_\zeta \in T_f$ for every $\zeta \in P$.

Then the mapping $\iota : \zeta \mapsto (\tau_\zeta^{-1}, \zeta)$ is a homomorphism from P onto a discrete cocompact subgroup of G_f . With the same proof as in [12, Prop. 7.2] that (5.13) induces a T -equivariant diffeomorphism $G \times_H M_h \rightarrow M$, we obtain that the mapping $((t_f, \zeta), x) \mapsto t_f \cdot e^{L_\zeta}(x) : G_f \times M_h \rightarrow M$ induces a T -equivariant diffeomorphism α_f from $M_f \times M_h$ onto M . Let c_h be the t_h -component of c in $t = t_h \oplus t_f$. Let π_f, π_h be the projection form $M_f \times M_h$ onto the first and the second factor, respectively. The symplectic form $\alpha_f^* \sigma$ on $M_f \times M_h$ is equal to $\pi_f^* \sigma_f + \pi_h^* \sigma_h$, and the symplectic form σ_f on M_f is given, according to (5.11) with $c_h = 0$, by

$$(\sigma_f)_b(\delta b, \delta' b) = \delta \zeta(\delta' t) - \delta' \zeta(\delta t). \quad (5.14)$$

Here $b = (t, \zeta) \iota(P) \in G_f/\iota(P)$, the tangent vectors $\delta b = (\delta t, \delta \zeta)$ and $\delta' b = (\delta' t, \delta' \zeta)$ are elements of $t_f \times N$. It follows that (M, σ, T) is T -equivariantly symplectomorphic to $(M_f, \sigma_f, T_f) \times (M_h, \sigma_h, T_h)$, in which (M_f, σ_f, T_f) is a compact connected symplectic manifold with a free symplectic action T_f -action. Here $t \in T$ acts on $M_f \times M_h$ by sending (x_f, x_h) to $(t_f \cdot x_f, t_h \cdot x_h)$, if $t = t_f t_h$ with $t_f \in T_f$ and $t_h \in T_h$.

Then the classification of symplectic toric manifolds [10] of Delzant implies that M_h is a 2-sphere equipped with an S^1 -action by rotations about the vertical axis and endowed with a rotationally invariant symplectic form. On the other hand M_f is diffeomorphic to $T \simeq \mathbb{T}^2$, since S^1 does not act freely on a surface non-diffeomorphic to T , which in turn implies that M is of the form given in part 2) of the statement.

Case 3.3.⁹ Suppose that all T -orbits are Lagrangian 2-tori and that T acts freely. That (M, σ) is as in part i) of the statement for a unique choice of the ingredients therein, follows from Theorem 5.6 once we show that the model $G \times_H M_h$ of (M, σ) in Claim 1.1 is of the form $(T \times \mathfrak{t}^*)/\iota(P)$ with the T -actions and symplectic form in part i), which we do next. This unique choice of ingredients is given by Definition 5.5. Since the T -action is free, $T_h = \{e\}$ and hence, by (5.9), $H = \{(t, \tau_\zeta) \mid \xi \in P, t\tau_\zeta = 0\} = \iota(P)$ and $\mathfrak{t}_h = \{0\}$, where ι was given in part i) of the statement. Also $G = T \times \mathfrak{t}^*$, c.f. (5.4). Because $T_h = \{1\}$, the Delzant submanifold M_h may be chosen to be any point $x \in M$, and there is a natural T -equivariant symplectomorphism

$$G \times_H M_h \rightarrow G/H \times \{x\} \rightarrow G/H = (T \times \mathfrak{t}^*)/\iota(P),$$

where $(T \times \mathfrak{t}^*)/\iota(P)$ is equipped with the symplectic form (5.14), but considering that $\delta b = (\delta t, \delta \zeta)$, $\delta' b = (\delta' t, \delta' \zeta) \in \mathfrak{t} \times \mathfrak{t}^*$. This expression for the symplectic form can be obtained by simplifying expression (5.11) according to σ_h, c_h trivial.¹⁰

⁹ The approach to the proofs of case 3.2 and 3.3 are different. In Case 3.3, the Delzant manifold M_h is trivial, so the proof is easily obtained as a particular case of the model $G \times_H M_h$. In Case 3.3 this approach does not lead to a product $M_f \times M_h$ directly, and hence why we used the same proof method as in Claim 1.1 but did not specialize. Although if $M_f := G_f/\iota(P)$ and $H^\circ := T_h \times \{1\}$, the mapping $g \mapsto g H^\circ$ defines an isomorphism from G_f onto the group G/H° , and an isomorphism from $\iota(P)$ onto H/H° , which leads to an identification of M_f with G/H , $G \times_H M_h$ is not even T -equivariantly diffeomorphic to $G/H \times M_h$ with the induced action.

¹⁰It follows that for any $x \in M$, the mapping

$$(t, \zeta) \mapsto t e^{L_\zeta}(x) : T \times \mathfrak{t}^* \rightarrow M,$$

Case 3.4. Suppose that all T -orbits are symplectic 2-tori. Then it follows from Theorem 4.33 that (M, σ) is T -equivariantly symplectomorphic to the symplectic T -manifold given in part ii) of the statement for a unique choice of ingredients, and this unique list is given in Definition 4.27 with $f = \mu$ and $\mu_h = f_h$.

This concludes the proof of Theorem 5.4. \square

5.3 Corollaries of Theorem 5.4

In the statement of Theorem 5.4, $(T \times \mathfrak{t}^*)/\iota(P)$ is a principal T -bundle over the torus \mathfrak{t}^*/P . Palais and Stewart [44] showed that every principal torus bundle over a torus is diffeomorphic to a nilmanifold for a two-step nilpotent Lie group. We have given an explicit description of this nilmanifold structure in Theorem 5.4 i). Theorem 5.4 also implies the following results.

Theorem 5.7. *The only symplectic 4-dimensional manifold equipped with a non-locally-free and non-Hamiltonian effective symplectic action of a 2-torus is, up to equivariant symplectomorphisms, the product $\mathbb{T}^2 \times S^2$, where $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$ and the first factor of \mathbb{T}^2 acts on the left factor by translations on one component, and the second factor acts on S^2 by rotations about the vertical axis of S^2 . The symplectic form is a positive linear combination of the standard translation invariant form on \mathbb{T}^2 and the standard rotation invariant form on S^2 .*

Proof. Since the T -action is not Hamiltonian, case 1) in the statement of Theorem 5.4 cannot occur. Since the action is non-locally-free, there are one-dimensional or two-dimensional stabilizer subgroups, (M, σ) cannot be as in item 3) or 4): in item 3) the stabilizers are all trivial, and in item 4) the stabilizers are finite groups. \square

Theorem 5.8. *Let (M, σ) be a non-simply connected symplectic 4-manifold equipped with a symplectic non-free action of a 2-torus T and such that M is not homeomorphic to $T \times S^2$, then (M, σ) is of the form given in part ii) of the statement of Theorem 5.4, for some $m \geq 1$.*

Proof. This follows by Theorem 5.4 by the fact that Delzant manifolds are simply connected. Indeed, every Delzant manifold can be provided with the structure of a toric variety defined by a complete fan, cf. Delzant [10] and Guillemin [17, App. 1], and Danilov [9, Th. 9.1] observed that such a toric variety is simply connected. The argument is that the toric variety has an open cell which is isomorphic to \mathbb{C}^n , of which the complement is a complex subvariety of complex codimension one. Therefore any loop can be deformed into the cell and contracted within the cell to a point. \square

Remark 5.9 The reasons because of which we have imposed that the torus T is 2-dimensional and (M, σ) is 4-dimensional in Theorem 5.4 are

induced by the mapping (5.13), gives a T -equivariant symplectomorphism between $(T \times \mathfrak{t}^*)/\iota(P)$ and (M, σ) .

- i) There does not exist a classification of n -dimensional smooth orbifolds if $n > 1$.
- ii) In dimensions greater than 2, the symplectic form is not determined by a single number (Moser's theorem).
- iii) If M is not 4-dimensional, and T is not 2-dimensional, then there are many cases where not all of the torus orbits are symplectic, and not all of the torus orbits are isotropic. Other than that Theorem 5.4 may be generalized. Let (M, σ) be a compact connected symplectic $2n$ -dimensional manifold equipped with an effective symplectic action of a torus T and suppose that one of the following two conditions hold:
 - 1. There exists a T -orbit of dimension $\dim T$ which is a symplectic submanifold of (M, σ) .
 - 2. There exists a principal T -orbit which is a coisotropic submanifold of (M, σ) .

Then such symplectic manifolds with T -actions are classified analogously to Theorem 5.4, but in weaker terms (e.g. involving an n -dimensional orbifold as in item i) above instead of the first two items of Definition 4.31).

□

References

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Alvaro Pelayo

Department of Mathematics, University of Michigan
2074 East Hall, 530 Church Street, Ann Arbor, MI 48109–1043, USA
e-mail: apelayo@umich.edu